# Polynomials

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# 1 Introduction

## 1.1 Definition

**Definition 1.** A polynomial is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $x$ is a variable and $a_i$, for $i = 0, 1, 2, \ldots, n$ are constants.

We call $n$ the degree of the polynomial. *Roots* are values of $x$ such that when substituted, the expression equals zero. Often as shorthand, we will denote polynomials as functions in $x$, like $p(x)$.

## 1.2 Polynomial Arithmetic

Polynomials behave much like integers in our ability to add and multiply them. For addition, adding is a simple matter of combining like terms. Multiplication occurs similarly. More involved is *polynomial division*.

**Definition 2.** Let $P(x)$ and $D(x)$ be two polynomials. Then there exist polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = Q(x) \cdot D(x) + R(x).$$

In analogy with integer division, $Q(x)$ is the quotient and $R(x)$ is the remainder.

So, given two polynomials, how do we find the quotient and remainder? The process is actually quite similar to integer division!

**Example 3.** Divide $x^4 + 2x^2 + 1$ by $x + 1$.

*Solution.*

$$
\begin{array}{rrrrrrr}
 & & x^3 & -\,x^2 & +\,3x & -\,3 \\
\cline{2-6}
x+1 \big) & x^4 & & +\,2x^2 & & +\,1 \\
 & -\,x^4 & -\,x^3 & & & \\
\cline{2-3}
 & & -\,x^3 & +\,2x^2 & & \\
 & & x^3 & +\,x^2 & & \\
\cline{3-4}
 & & & 3x^2 & & \\
 & & & -\,3x^2 & -\,3x & \\
\cline{4-5}
 & & & & -\,3x & +\,1 \\
 & & & & 3x & +\,3 \\
\cline{5-6}
 & & & & & 4
\end{array}
$$

We thus have that

$$x^4 + 2x^2 + 1 = \left(x^3 - x^2 + 3x - 3\right)(x+1) + 4.$$

□

**Problem 4.** Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2+x+1$, the remainder is $x+2$, and when $P(x)$ is divided by the polynomial $x^2+1$, the remainder is $2x+1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

**Walkthrough 1.** Don't overcomplicate it!

**(a)** First, verify that $P(x)$ cannot have a degree of 2 or lower. Now, consider the case where $P(x)$ has a degree of 3.

**(b)** $P(x) = Q_1(x)(x^2+x+1) + x + 2 = Q_2(x)(x^2+1) + 2x + 1$. $P(x)$ can also be expressed as $ax^3 + bx^2 + cx + d$. What does this imply about the degree of $Q_1$ and $Q_2$?

**(c)** $Q_1(x) = m_1x + n_1$, $Q_2(x) = m_2x + n_2$. Can you see why $m_1 = m_2 = a$? Multiply out, and match like terms.

### 1.3 Remainder Theorem

**Theorem 5.** Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when we divide the polynomial $f(x)$ by $x-a$. Then $r(a) = f(a)$.

*Proof.* We have that $f(x) = (x-a)q(x) + r(x)$. Suppose that $x = a$. Then our $(x-a)q(x)$ term is equal to zero, and $f(a) = r(a)$. □

Note that the degree of $r(x)$ is zero, so it is a constant.

**Problem 6.** Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x-19$, the remainder is 99, and when $P(x)$ is divided by $x-99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x-19)(x-99)$?

**Walkthrough 2.** Don't blindly use the remainder theorem; think carefully about what it implies and its underlying principles.

**(a)** What's the degree of the remainder? What form will the remainder take?

**(b)** Write out the expression as $P(x) = (x-19)(x-99)Q(x) + R(x)$. What happens if you plug in $P(19)$ or $P(99)$?

**(c)** You should have a system of equations that can be easily solved.

**Problem 7.** Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

**Walkthrough 3.** **(a)** Factor $x^3 - x$.

**(b)** Use the remainder theorem for each of the individual factors.

**(c)** The remainder takes the form of $ax^2 + bx + c$.

### 1.4 Coefficients

Often you'll want to find the sum of coefficients of a polynomial. To do this, note that you can simply substitute in $x = 1$.

**Problem 8.** For all values of $x$,

$$x^{19} + 11x^{17} - 3x^{13} + 9x^7 + 17x^3 = (2x^5 - 3x^4 + 7x^3 + 5x^2 - x - 3)P(x)$$

where $P(x)$ is a polynomial in $x$. Find the sum of all the coefficients of $P(x)$.

Now, a similar problem, but with a caveat!

**Example 9.** For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* Finding $\sum_{i=0}^{50} a_i$ is simple, as it is simply equal to $Q(1)$. But it seems impossible to find the sum of the absolute value of all the coefficients! What we have to do is identify which coefficients are negative. Note that $P(x^n) = 1 - \frac{1}{3}x^n + \frac{1}{6}x^{2n}$, so in any of the polynomials that we're multiplying together, the only negative factors come from the $-\frac{1}{3}x^n$ terms. When we multiply together all of the polynomials, terms with negative coefficients will have an odd number of factors from $-\frac{1}{3}x^n$. Now, observe that these terms have odd degrees, and we will always be multiplying an odd number of them, and the sum of degrees is odd. In addition, any other terms we will multiply will have an even degree. Thus we can conclude that *every term with a negative coefficient has odd degree.* Thus, we can simply evaluate $Q(-1) = \left(\frac{3}{2}\right)^5$. □

### 1.5 Factor Theorem

A polynomial $d(x)$ evenly divides a polynomial $f(x)$ if the remainder is 0 when we divide $f(x)$ by $d(x)$. We then call $d(x)$ a *factor* of $f(x)$.

**Theorem 10.** Let $p(x)$ be a polynomial.

**(a)** If $x - a$ is a factor of $p(x)$, then $p(a) = 0$.

**(b)** If $p(a) = 0$, then $x - a$ is a factor of $p(x)$.

*Proof.* **(a)** $p(x) = (x - a)q(x)$. If we substitute in $x = a$, $p(a) = 0$.

**(b)** By the remainder theorem, when we divide $p(x)$ by $x - a$, the remainder is $p(a) = 0$. Thus $x - a$ is a factor of $p(x)$.

□

With this theorem, we have a first glimpse of a relation between the roots and the coefficients of a polynomial. A second theorem gives us even more powerful tools to find roots.

**Theorem 11** (Fundamental Theorem of Algebra)**.** Every one-variable polynomial of degree $n$ has exactly $n$ complex roots (not necessarily distinct).

This assertion lets us derive a very useful result.

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

**Problem 12.** The polynomials

$$x^3 + 5x^2 + px + q = 0$$

and

$$x^3 + x^2 + px + r = 0$$

have exactly two roots in common, so each cubic has a root they do not have in common. Find the sum of the two roots they do not have in common.

**Walkthrough 4.** When problems talk about common roots, the best way of attack is usually through decomposing into individual factors.

**(a)** Let $x^3 + 5x^2 + px + q = (x - r)(x - s)(x - a)$ and $x^3 + x^2 + px + r = (x - r)(x - s)(x - b)$.

**(b)** Subtract the two polynomials. Can you derive anything about the two common roots?

# 2 Roots

## 2.1 Vieta's Formulas

Recall that any polynomial of degree n can be expressed as

$$a_n(x - r_1)(x - r_2)\cdots(x - r_n).$$

What happens when we expand and multiply this out? The coefficient of $x^n$ will simply be $a_n$. For the coefficient of $x^{n-1}$, $a_{n-1}$, we can select a single $-r_i$ term. Thus $a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n)$. If we call the sum of the roots $r_1 + r_2 + \cdots + r_n = S_1$, the *first symmetric sum*, we have that $S_1 = -\frac{a_{n-1}}{a_n}$. For $a_{n-2}$, after selecting $n-2$ $x$ terms, we can select another 2 $-r_i$ terms. There are $\binom{n}{2}$ ways to do this, and we have that

$$r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n = S_2 = \frac{a_{n-2}}{a_n}.$$

Further coefficients and symmetric sums can be evaluated similarly. When we get to $a_0$, we find that the product of all the roots is $\frac{a_0}{a_n} = (-1)^n r_1 r_2 \ldots r_n$. These are called *Vieta's formulas*, and show up very frequently in math competitions.

**Problem 13.** The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism. A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

**Walkthrough 5.** A simple introduction to demonstrate the utility of Vieta's formulas.

**(a)** Let the original lengths be $r$, $s$, and $t$. Find an expression for the volume of the new box and expand it out.

**(b)** Can you see the symmetric sums? We already know them through the coefficients of the polynomial.

For this problem, we can also use an approach that avoids Vieta's.

*Solution.* $10x^3 - 39x^2 + 29x - 6 = 10(x - r)(x - s)(x - t)$. If we plug in $x = 0$, we get that $-10rst = -6$, and that the original volume of the box is $\frac{3}{5}$. Now, if we plug in $x = -2$, we have that $-80 - 152 - 58 - 6 = 10(-2 - r)(-2 - s)(-2 - t)$, or simplifying,

$$(r + 2)(s + 2)(t + 2) = 148/5.$$

□

Note that in neither case do we have to solve for the roots! This saves us tremendous effort.

**Problem 14.** All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers. What is the value of $B$?

**Walkthrough 6.** Combined with other information, Vieta's can help us derive the roots.

**(a)** What's the sum of the roots? What's their product? Note that they have to be positive integers. This is sufficient to derive the roots!

**(b)** The problem asks for the coefficient of $B$. What symmetric sum does this correspond to?

**Problem 15.** For certain real numbers $a$, $b$, and $c$, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

**Walkthrough 7.** The coefficients being variables hints at a Vieta's approach.

**(a)** Let the roots of $g$ be $p$, $q$, and $r$. Let the additional root of $f$ be $s$. Compute all of the symmetric sums.

**(b)** Look at $pqr + pqs + prs + qrs$. We know $pqr$. What common factor do the other three terms share? Factor it out.

**(c)** Now that you know the value of $s$, you should be able to derive $a$ as well. Look at $pq + qr + pr + ps + qs + rs$. Can we do something like before?

We conclude with a very difficult problem involving Vieta's.

**Problem 16.** For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* To start, we should expand out the polynomial to

$$(x - z_{1_1})(x - z_{1_2})(x - z_{1_3}) \cdots (x - z_{673_1})(x - z_{673_2})(x - z_{673_3}).$$

We seek the second symmetric sum. This sum will consist of two kinds of terms. The first case is where the roots are distinct, and their sum is

$$9 \cdot \sum_{1 \le j < k \le 673} z_j z_k.$$

The factor of 9 comes because for each $z_i$ term, there are 3 other terms that are identical and that can also be chosen. The second case is where both of the roots are equal. Their sum will be

$$3 \cdot \sum_{i=1}^{673} z_i^2.$$

The factor of three is because within $z_{i_1}$, $z_{i_2}$, and $z_{i_3}$, there are three ways to choose a pair. We have that

$$9 \cdot \sum_{1 \le j < k \le 673} z_j z_k \; + \; 3 \cdot \sum_{i=1}^{673} z_i^2 = 19.$$

Our next step is to determine the sum of the squares of the roots. We know that $\cdot \sum_{i=1}^{673} z_i = -\frac{20}{3}$. Squaring this expression, we have that

$$\sum_{i=1}^{673} z_i^2 + 2 \cdot \sum_{1 \le j < k \le 673} z_j z_k = \frac{400}{9}.$$

Then,

$$3 \cdot \sum_{i=1}^{673} z_i^2 = \frac{400}{3} - 6 \cdot \sum_{1 \le j < k \le 673} z_j z_k.$$

Substituting this in, we have that

$$\frac{400}{3} + 3 \cdot \sum_{1 \le j < k \le 673} z_j z_k = 19.$$

We finally have that

$$\left| \sum_{1 \le j < k \le 673} z_j z_k = 19 \right| = \boxed{\frac{343}{9}}.$$

□

## 2.2 Irrational and Complex Roots

When polynomials have irrational or complex roots, while we develop new techniques, many of our previous results are still valid.

**Theorem 17.** Let $d = a + bi$, where $i = \sqrt{-1}$ and $a$ and $b$ are real. If $d$ is a root of polynomial $f(x)$ which has real coefficients, then $\overline{d}$ is also a root of $f(x)$. $\overline{d}$ denotes the complex conjugate of $d$.

In fact, complex roots always come in pairs. We find it useful to introduce the idea of *multiplicity*. Recall that polynomials can be expressed as $a_n(x - r_1)(x - r_2) \cdots (x - r_n)$. The roots have no requirement to be distinct, and we say that a root which occurs in this product $m$ times has multiplicity $m$.

**Problem 18** (2014 HMMT)**.** Find all real numbers $k$ such that $x^4 + kx^3 + x^2 + 4kx + 16 = 0$ is true for exactly one real number $x = r$. Enter all the possible values of $k$, separated by commas.

*Solution.* □

**Problem 19.** Find the sum of all complex values of $a$, such that the polynomial $x^4 + (a^2 - 1)x^2 + a^3$ has exactly two distinct complex roots.

*Solution.* There will be a total of 4 roots. For there to be two distinct complex roots, there are a number of cases we can consider. If one root is $a + bi$ where $a$ and $b$ are nonzero, then $a - bi$ is also a root, and these are our two distinct roots, each with multiplicity two. The polynomial is thus $(x^2 - 2ax + a^2 + b^2)^2$. A quick check reveals that $a$ must be zero. We are left with $x^4 + 2bx^2 + b^4$. $a^2 - 1 = 2b$, and $b^4 = a^3$. We then have $16(a^2 - 1)^4 = a^3$. This is a degree 8 polynomial in terms of $a$, and we can tell that the coefficient of $a^7$ is zero. So the sum from this case is zero. The alternative case is where we have 2 real roots (real numbers are also complex!); call these $r$ and $s$. Their multipkicities must sum to 4. Consider WLOG that the polynomial is $(x - r)^3(x - s)$. The sum of the roots must be zero, so $s = -3r$. We then have that the polynomial is $x^4 + (3r^2 - 9r^2)x^2 - (r^3 - 9r^3)x - 3r^4$. We get that $r^3$ must be zero, so this solution is invalid. The only other possible case is $(x^2 - (r+s)x + rs)^2 = x^4 - 2(r-s)x^3 + (r^2 + 2rs + s^2 + 2rs)x^2 - 2rs(r+s)x + r^2s^2$. We can immediately see that $r = -s$, so we can simplify this to $x^4 - 2r^2x^2 + r^4$. This gives $a^2 - 1 = -2r^2$ and $a^3 = r^4$. We then have that $4a^3 = (a^2 - 1)^2 = a^4 - 2a^2 + 1$. The sum of possible values for $a$ is thus 4. □

**Problem 20.** The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a} + \sqrt[3]{b} + 1}{c}$, where $a$, $b$, and $c$ are positive integers. Find $a + b + c$.

**Walkthrough 8.** This requires knowing a certain identity!

**(a)** Note that $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$.

**(b)** After taking the cube root, you have a degree 1 equation to solve.