Probability & Expected Value

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"Heads or tails?"

1 Introduction

In many ways, probability is just counting. If all outcomes are equally likely, the probability of an event X happening is simply characterized by the equation

$$\mathbb{P}(X) = \frac{\text{number of outcomes with where } X \text{ happens}}{\text{total number of possibilities}}.$$

For example, the probability of flipping a coin and flipping heads is $\frac{1}{2}$, since there is 1 outcome where the coin shows heads and 2 total possible outcomes.

2 Basic Probability

2.1 Introduction

As we briefly went over above, the probability something happens is just the chance that something happens. For example, you have probably heard of things having a "50-50" chance of happening - that's probability. Or if in the weather report, there's a 90% chance of rain - that's also probability. So how does probability work?

2.2 Compounded Probability

One equation that fuels most probability problems is the following equation. If events X and Y are independent, then

$$P(X + Y) = P(X) * P(Y).$$

For example, if the probability of flipping heads on a coin is $\frac{1}{3}$, then the probability of flipping heads on two consecutive flips would be $\frac{1}{3} * \frac{1}{3}$, or $\frac{1}{9}$. Similarly, you can also add probabilities. If events X and Y are independent and mutually exclusive (meaning that they cannot happen at the same time), then

$$P(X \text{ or } Y) = P(X) + P(Y).$$

2.3 Conditional Probability

Similarly, a type of probability is also conditional probability, the probability an event X happens given that Y happens. This type of probability is usually given by the equation

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)},$$

where $P(X \mid Y)$ denotes the probability that X happens given that Y happens and $P(X \cap Y)$ is the probability that both X and Y happen. For example, consider the following problem;

2.3.1 Example - Children

The Smiths are having two children. Given that at least one of the children is a girl and each baby is equally likely to be a boy or a girl, what is the probability that both children are girls?

2.3.2 Solution - Children

Notice that the answer is NOT $\frac{1}{2}$. Instead, we use the equation given. The probability that both children are girls is $\frac{1}{4}$, and the probability that at least one of the children is a girl is $\frac{3}{4}$. Dividing, we have that the probability that both of the children are girls given that at least one of the children is a girl is $\frac{\frac{1}{4}}{\frac{3}{4}}$, or $\frac{1}{3}$.

3 Probability with Unknown Variables

In your quest to solve more problems, you may encounter problems where the ending probability is given, and you must find how many of x thing there are, or how many of another y thing there are in a group of z, etc. In this case, we can use our knowledge of how probability works to solve these problems in the following examples.

3.1 Examples

3.1.1 Example - Marbles in a Bag

There are 7 red marbles in a bag and some number of blue marbles. If we draw two marbles with replacement, and the probability both marbles drawn are red is $\frac{1}{9}$, how many blue marbles are there?

3.1.2 Solution - Marbles in a Bag

Let there be b blue marbles in a bag. Notice that each time, the probability of drawing a red marble is $\frac{7}{b+7}$, so we have that the probability of drawing a red marble both times is $(\frac{7}{b+7})^2$, which is also $\frac{1}{9}$. Solving the equation, we find that $\frac{7}{b+7} = \frac{1}{3}$ (which is the square root of $\frac{1}{9}$), and cross-multiplying gives us that $b = 14$. Therefore there are 14 blue marbles in the bag.

As we can see, in problems like these, the key is to use variables in place of numbers and use these variables to solve equations.

4 Fixing Variables to Deal With Large Values

We begin this section with the following example problem.

4.1 Example - Pairings

In a class of 30 students, the teacher randomly pairs the students into 15 disjoint pairs. One student, Alex, wants to be paired with his friend Richard. What is the probability Alex and Richard are paired together?

Of course, the most "obvious" way to solve this problem is to count the number of ways to pair off the students, and the number of pairings there are where Alex and Richard are together. But through some calculations, we find that the number of total pairings is

$$\frac{1}{15!} * \binom{30}{2}\binom{28}{2}\binom{26}{2}\cdots\binom{4}{2}\binom{2}{2},$$

which as we can already see, is a pretty big number.

4.1.1 "Better" Solution - Pairings

Notice that Alex must always be paired with someone - and Alex can be paired with each of the other 29 people with equal probability. However, Richard is exactly one of those 29 people, meaning that the probability that Alex and Richard are paired together is $\frac{1}{29}$.

As we can see, focusing on ONE important aspect of a problem can greatly simplify your computations. As with many problems in Counting & Probability, we find that some problems will be straight-up bash, and others require clever manipulations to turn the problem from large problems to smaller, manageable problems. If you have done AoPS's Alcumus before, you may recall these problems are listed under the "Think About It!" section in the Counting & Probability topics. For example;

4.2 Example - Flipping Cards (Alcumus)

Carson flips over the cards of a standard 52-card deck one at a time. What is the probability that he flips over the ace of spades before any face card (jack, queen or king)?

4.2.1 Solution - Flipping Cards (Alcumus)

Notice that here, all of the other cards are irrelevant. Focusing on the face cards and the ace of spades only (there are 13 total of these cards), we find that each possible order Carson draws these in has equal probability, therefore all 13 of these cards have equal probability of being drawn first out of the thirteen. Therefore, the probability that Carson draws the ace of spades before any face card is exactly $\frac{1}{13}$.

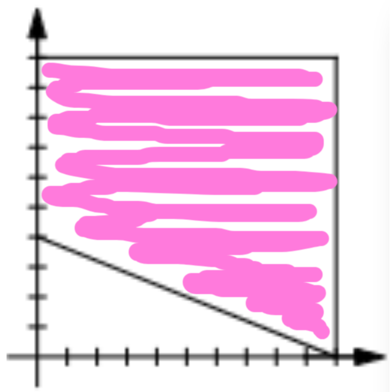
5 Geometric Probability

Another way to solve counting and probability problems is geometrically, by drawing out a rectangle or square of some sort to obtain a probability. Consider the following example;

5.1 Example - Points in a Plane (Alcumus)

A point with coordinates $(x,\ y)$ is randomly selected such that $0 \le x \le 10$ and $0 \le y \le 10$. What is the probability that the coordinates of the point will satisfy $2x + 5y \ge 20$? Express your answer as a common fraction.

In geometric probability, this is one of the problems with a more obvious application of geometric probability.



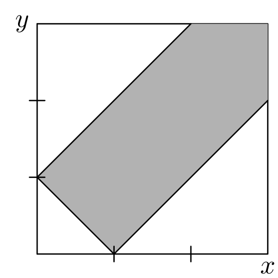
The point can be randomly selected anywhere inside the big square, which has area $10 \cdot 10 = 100$. The point satisfies the given inequality if it lies within the shaded region (the diagonal side of the shaded region is a segment of the line $2x + 5y = 20$). We will find its area by subtracting the area of the non-shaded region from the area of the square. The non-shaded region is a triangle with base of length 10 and height of length 4, so its area is $\frac{10 \cdot 4}{2} = 20$. The area of the shaded region is then $100 - 20 = 80$. So the probability that the point falls within the shaded region is $80/100 = \boxed{4/5}$.

5.2 Example - Triangles and Randomly Chosen Side Lengths (Alcumus)

Two numbers, x and y are selected at random from the interval $(0, 3)$. What is the probability that a triangle with sides of length 1, x, and y exists?

5.2.1 Solution - Triangles and Randomly Chosen Side Lengths (Alcumus)

If a triangle with sides of length 1, x, and y exists, the triangle inequality must be satisfied, which states that $x + y > 1$, $1 + x > y$, and $1 + y > x$. We can draw a plane with x and y axes and shade in the area where all of these inequalities are satisfied.



The total area of the square is $3^2 = 9$. The area of the unshaded region is $2^2 + 1/2 = 9/2$. Thus, the shaded area is $9/2$ and the probability that such a triangle exists is $(9/2)/9 = \boxed{1/2}$.

6 Expected Value

Another aspect of combinatorics is the idea of expected value - in other words, the "average" value of an outcome. The EV of a variable is computed by taking the sum of all possible values times their probabilities. In equation form;

$$E(X) = \sum p_i x_i,$$

where p_i is the probability of the value x_i occuring.

6.1 Example - Coin Toss (Alcumus)

A certain coin is weighted such that the chance of flipping heads is $\frac{1}{3}$ and the chance of flipping tails is $\frac{2}{3}$. Suppose that we win \$3 if we flip a heads on a coin toss, but lose \$2 if we flip tails. What is the expected value, in dollars, of our winnings after one flip? Express your answer as a common fraction.

6.1.1 Solution - Coin Toss (Alcumus)

In one flip, we have a 1/3 chance of getting heads and winning 3 dollars, and a 2/3 chance of getting tails and losing 2 dollars. So the expected value of one flip is $E = \frac{1}{3}(\$3) + \frac{2}{3}(-\$2) = \boxed{-\frac{1}{3}}$.

We will continue our exploration of expected value problems through a slightly more challenging problem.

6.2 Example - Fair Price of a Bet (Alcumus)

I draw a card from a standard 52-card deck. If I draw an Ace, I win 1 dollar. If I draw a 2 through 10, I win a number of dollars equal to the value of the card. If I draw a face card (Jack, Queen, or King), I win 20 dollars. If I draw a $\clubsuit$, my winnings are doubled, and if I draw a $\spadesuit$, my winnings are tripled. (For example, if I draw the $8\clubsuit$, then I win 16 dollars.) What would be a fair price to pay to play the game? Express your answer as a dollar value rounded to the nearest cent. (In other words, what is the expected value of your winnings when playing this game?)

6.2.1 Solution - Fair Price of a Bet (Alcumus)

Let E_1 be the expected winnings if a $\heartsuit$ or $\diamondsuit$ is drawn. Since the probability that any particular rank is drawn is the same for any rank, the expected value is simply the average all the winnings for each rank, so

$$E_1 = \frac{1}{13}(\$1 + \$2 + \cdots + \$10 + (3 \times \$20)) = \$\frac{115}{13}.$$

Let E_2 be the expected winnings if a $\clubsuit$ is drawn and E_3 the expected winnings if a $\spadesuit$ is drawn. Since drawing a $\clubsuit$ doubles the winnings and drawing a $\spadesuit$ triples the winnings, $E_2 = 2E_1$ and $E_3 = 3E_1$. Since there is an equal chance drawing each of the suits, we can average their expected winnings to find the overall expected winnings. Therefore the expected winnings are

$$E = \frac{1}{4}(E_1 + E_1 + E_2 + E_3) = \frac{1}{4}(7E_1) = \$\frac{805}{52},$$

or about $\boxed{\$15.48}$, which is the fair price to pay to play the game.

7 Combined Walkthroughs

-Provide walkthroughs and examples that go over all topics covered in the handout

Problem 1 (Fall 2021 AMC 10A)**.** When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

Problem 2 (Fall 2021 AMC 10A)**.** Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

Problem 3 (Fall 2021 AMC 10B)**.** Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

Problem 4 (2019 AMC 10A)**.** Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

Problem 5 (Fall 2021 AMC 10B)**.** In a particular game, each of 4 players rolls a standard 6-sided die. The winner is the player who rolls the highest number. If there is a tie for the highest roll, those involved in the tie will roll again and this process will continue until one player wins. Hugo is one of the players in this game. What is the probability that Hugo's first roll was a 5, given that he won the game?

Problem 6 (Alcumus)**.** The cards of a standard 52-card deck are dealt out in a circle. What is the expected number of pairs of adjacent cards which are both black?

Walkthrough 1. A basic probability problem.

(a) Determine the respective probabilities for rolling an even number and an odd number for one dice.

(b) What possibilities lead to the sum being even?

Walkthrough 2. A more involved problem requiring techniques from combo.

(a) Denote the bins A, B, C, D, E. Every possibility is thus a 20 letter string consisting of these five letters.

(b) Let's say that there are 4 As, 4 Bs, 4 Cs, and so on. How many different strings can you make?

(c) Now consider if there are 3 As, 5 Bs, and 4 of every other letter. How many strings can you make now?

(d) We've undercounted. To correct this, we need to account for selecting two bins to have 3 and 5 balls in them.

Walkthrough 3. Carefully account for the possible cases.

(a) In this case, it'll likely be easier to count the cases such that the product is not divisible by 4.

(b) What's the probability that the product is odd?

(c) What's the probability that the product is only divisible by 2?

Walkthrough 4. This problem brings together several concepts in probability.

(a) First consider the possibility that x and y are chosen uniformly from the interval. What's the probability that $|x - y| > \frac{1}{2}$? What's the occurrence of this case?

(b) Now look at the discrete possibilities, which are $(x, y) = (1, 0), (0, 1)$.

(c) Now consider the cases where x or y are 0 or 1, and the other variable is chosen from the interval.