Pythagorean Theorem and Synthetic Geometry

Richard Yang

June 7, 2023

1 Introduction

The Pythagorean Theorem is fundamental to geometry, and almost every geometry problem you see on the AMC and AIME exam will make use of it. First, we will introduce Power of a Point, a closely related technique that has immense utility with circles.

1.1 Proofs

Theorem (Power of a Point)**.** Consider a circle with center O and an arbitrary point P. We define the **power of** P as $OP^2 - r^2$, r being the radius of the circle.

- If ℓ is a line through P that intersects the circle at two distinct points X and Y, then
$$PX \cdot PY = \left|OP^2 - r^2\right|.$$
- If P is outside the circle and $\overline{PA}$ is tangent with A on the circle, then $PA^2 = OP^2 - r^2$. Note that this is a special case of the above.

We can prove this by using similarity and angle chasing. The converse is also true!

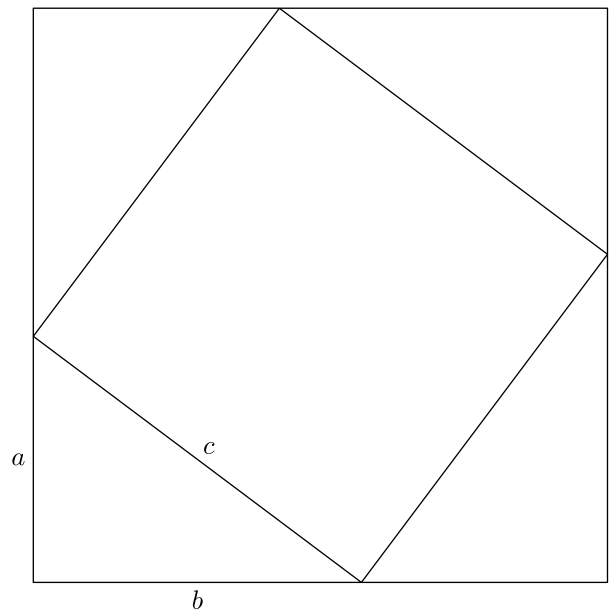
Theorem (Converse of PoP)**.** Let A, B, X, and Y be four distinct points in the plane. Let lines AB and XY intersect at point P. Suppose P lies in either both of the segments $\overline{AB}$ and $\overline{XY}$ or neither. If $PA \cdot PB = PX \cdot PY$, then A, B, X, and Y are concyclic.

This theorem allows us to find cyclic quadrilaterals based on length.

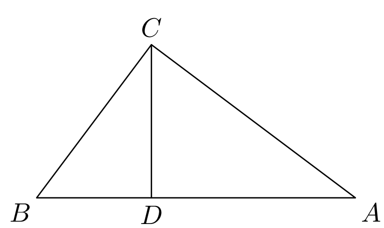
Power of a Point is extremely useful for problems involving circles, and is a basis for olympiad techniques like the radical axis. For our purposes, we can often use it in conjunction with the Pythagorean theorem in order to derive lengths.

Theorem (Pythagorean Theorem)**.** For any right triangle with legs length a, b and hypotenuse length c,

$$a^2 + b^2 = c^2.$$



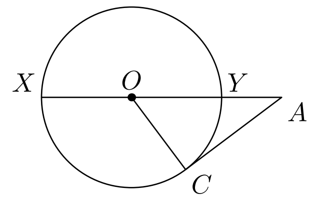
Nested Squares. Observe the above figure. We have a square with side length $a + b$, so it will have area $(a + b)^2 = a^2 + 2ab + b^2$. We can split the square into four triangles and one square. Each triangle has area $\frac{1}{2}ab$, so the total area of the triangles is $2ab$. The area of the inside square is therefore $a^2 + b^2$. Thus, $c^2 = a^2 + b^2$. $\square$



Similarity. Observe that $\triangle BCD$, $\triangle CAD$, and $\triangle BAC$ are all similar. Let $BC = a$, $CA = b$, and $BA = c$. The area of $\triangle BCD$ is proportional to a^2, so for some x, $[BCD] = a^2x$. Similarly, $[CAD] = b^2x$, and $[BAC] = c^2x$. But $[BAC] = [BCD] + [CAD]$, so $c^2x = a^2x + b^2x$. Thus,

$$a^2 + b^2 = c^2.$$

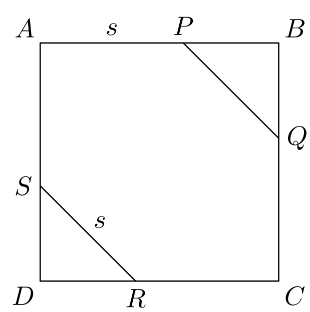
$\square$



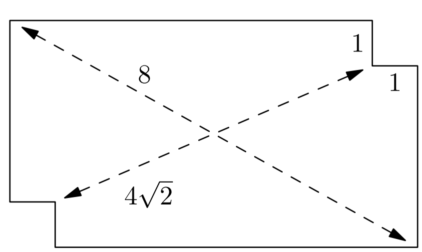
Power of a Point. Let $OX = OC = r$. As AC is tangent to circle O, by Power of a Point, we have $AC^2 = AY \cdot AX$. We know $AY = AO - r$ and $AX = AO + r$, so $AC^2 = (AO - r)(AO + r) = AO^2 - r^2$. Thus $AC^2 + r^2 = AO^2$. $\square$

1.2 Problems

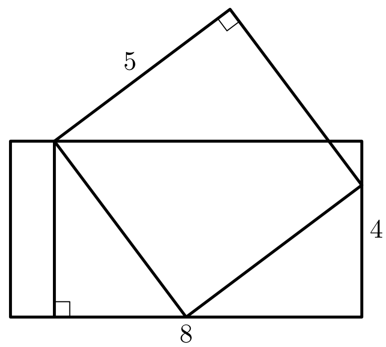
Problem 1 (2022 AMC 10 A)**.** Square $ABCD$ has side length 1. Point P, Q, R, and S each lie on a side of $ABCD$ such that $APQCRS$ is an equilateral convex hexagon with side length s. What is s?



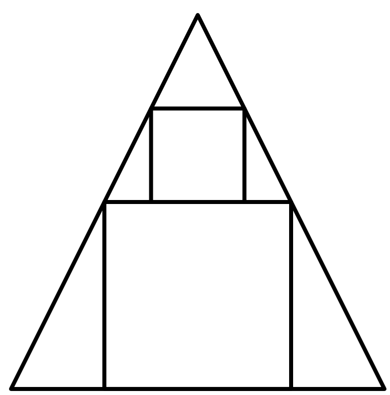
Problem 2 (2022 AMC 10 A)**.** Daniel finds a rectangular index card and measures its diagonal to be 8 centimeters. Daniel then cuts out equal squares of side 1 cm at two opposite corners of the index card and measures the distance between the two closest vertices of these squares to be $4\sqrt{2}$ centimeters, as shown below. What is the area of the original index card?



Problem 3 (2022 AMC 10 B)**.** The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



Problem 4. A square with side length 3 is inscribed in an isosceles triangle with one side of the square along the base of the triangle. A square with side length 2 has two vertices on the other square and the other two on sides of the triangle, as shown. What is the area of the triangle?



1.3 Walkthroughs

Walkthrough 1. Although simple, this problem demonstrates a common process for geometry problems. First we use the Pythagorean Theorem to find a relationship between lengths, then we use algebra to solve.

(a) Note that $PB = BQ = 1 - s$. How can we derive a length for PQ from this?

(b) With your expression from before, and knowing that $PQ = s$, solve for s.

Walkthrough 2. A more involved problem with more algebraic manipulation.

(a) Let the sides of the index card have length x and y. Draw the right triangle with hypotenuse $4\sqrt{2}$; what are the lengths of its legs?

(b) Expand $(x-2)^2 + y(-2)^2$. Do we already know the value of $x^2 + y^2$?

(c) You can get a xy term from $x + y$ by squaring it.

Walkthrough 3. This problem demonstrates how similarity and the Pythagorean theorem are often found together.

(a) So many similar right triangles! Do you notice anything special about their length ratio?

(b) It's easier to find the area of the square outside the rectangle, than subtract.

Walkthrough 4. More similarity!

(a) Let the base of the big triangle be $3 + 2x$. Derive a ratio between the altitude and the base of the isosceles triangle.

(b) Find the altitude of the triangle as a sum of the heights of the squares and the small triangle at the top.

(c) Knowing that the altitude of the big triangle should be $(2x+3) \cdot \frac{3}{2x}$, equate your expressions and solve for x.

2 Advanced Problems

2.1 Similarity

Problem 5 (Spring 2021 AMC 10)**.** Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

Solution. Observe that as $\triangle BCD$ is isosceles, CP is a perpendicular bisector. Next, as $\angle ABD = \angle BDC$, $\triangle ABD$ and $\triangle CDP$ are similar right triangles. In addition, $\angle AOD = \angle BOC$, so $\triangle ADO \sim \triangle CPO$. Let $CP = x$ and $DO = y$. By similarity, $AD = \frac{xy}{11}$. The ratio

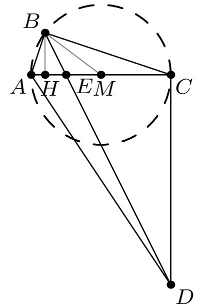
$$\frac{BD}{DA} = \frac{2y+22}{\frac{xy}{11}} = \frac{DP}{CP} = \frac{y+11}{x}.$$

We solve and get $y = 22$.

$$x = \sqrt{43^2 - (11+y)^2} = \sqrt{760},$$

so $DA = \frac{xy}{11} = 4\sqrt{190}$. Our final answer is $\boxed{194.}$ □

Problem 6. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90°, AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?



Solution. Observe that $\triangle BHE \sim \triangle DCE$. Let $HE = x$. We know $\frac{DC}{CE} = \frac{30}{15}$, so $BH = 2x$. Furthermore, $\triangle BHC \sim \triangle AHB$, so letting $AH = y$, we have

$$\frac{AH}{HB} = \frac{BH}{HC} = \frac{2x}{x+15} = \frac{y}{2x}.$$

So $y = \frac{4x^2}{x+15}$, and

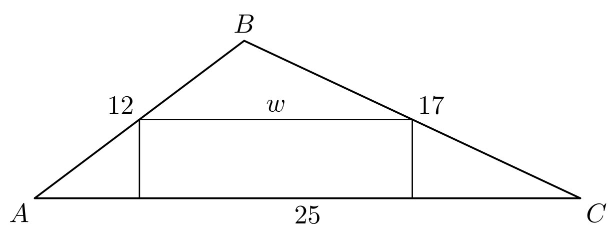
$$\frac{4x^2}{x+15} + x + 15 = 20.$$

We reduce this to the quadratic $x^2 + 2x - 15 = 0$ and obtain $x = 3$. The area of $\triangle ABC = x \cdot 20 = 60$, and the area of ACD is 300. Thus the total area is $\boxed{360.}$ □

Problem 7 (2015 AIME II)**.** Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2$$

Then the coefficient $\beta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution. We first try to determine the height of $\triangle ABC$. Let D be the foot of the perpendicular from B to AC. Let $AD = x$; then $DC = 25 - x$. We use the Pythagorean theorem to determine $BD^2 = 144 - x^2$, and that $289 = BD^2 + DC^2$. Thus

$$144 - x^2 + (25 - x)^2 = 144 - x^2 + x^2 - 50x + 625 = 289.$$

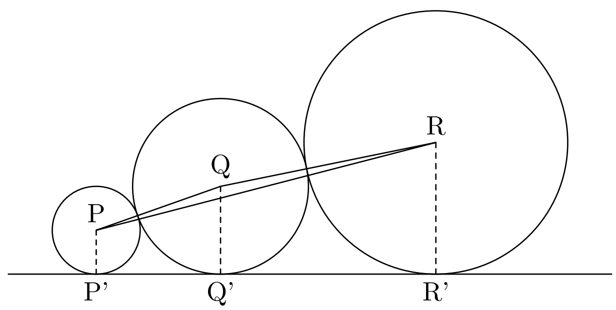
Simplifying, we get $x = \frac{48}{5}$. Thus, the length of BD is $\frac{36}{5}$. Now, $\triangle BPQ$ is similar to $\triangle BAC$, and the height of its altitude should be $\frac{w}{25} \cdot \frac{36}{5}$. Then the height of the rectangle is $\frac{36}{5}\left(1 - \frac{w}{25}\right)$. The area of $PQRS$ is thus $\frac{36}{5}w - \frac{36}{125}w^2$. The coefficient $\beta = \frac{36}{125}$, and our answer is $\boxed{161.}$ □

2.2 Circles

Problem 8 (2021 AMC 10 A)**.** Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C?

Solution. Let O be the center of the circle. The key observation is that $\angle ABO$ and $\angle ACO$ are 90 degrees. This means that $ABOC$ is a cyclic quadrilateral! AO is therefore the diameter of the circle that circumscribes $\triangle ABC$, and we obtain $AO = \sqrt{54+50} = 2\sqrt{26}$. The area of the circle is thus $\boxed{26\pi.}$ □

Problem 9 (2016 AMC 10 A)**.** Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR?



Solution. First we find $P'Q'$. Using the Pythagorean theorem, $P'Q'^2 + (2-1)^2 = (1+2)^2$, and we get $P'Q' = 2\sqrt{2}$. Similarly, $Q'R' = 2\sqrt{6}$. We can find the area of $\triangle PQR$ by summing the areas of $PQQ'P'$ and $QRR'Q'$, then subtracting the area of $PRR'P'$. Doing so, we get $\boxed{\sqrt{6}-\sqrt{2}.}$ □

Problem 10 (AIME 2015 II)**.** The circumcircle of acute $\triangle ABC$ has center O. The line passing through point O perpendicular to $\overline{OB}$ intersects lines AB and BC at P and Q, respectively. Also $AB = 5$, $BC = 4$, $BQ = 4.5$, and $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. We can use power of a point on Q. Let the circumradius be R; then $OQ^2 - R^2 = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$. In addition, $\triangle BOQ$ is a right triangle, so $BO^2 + OQ^2 = BQ^2$. Substituting, we get

$$R^2 + \frac{9}{4} + R^2 = \frac{81}{4}.$$

From this, we have $R = 3$. Using power of a point again, $R^2 - PO^2 = BP \cdot PA = BP(5 - BP)$. $PO^2 = BP^2 - R^2$ by the Pythagorean theorem, so $5BP = 18$. Our answer is thus $\boxed{23.}$ □