Mock Target

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1 Problem 1

Question: Richard has a jar of marbles containing 15 green marbles, 20 red marbles, and a certain number of yellow marbles. If $\frac{2}{7}$ of the marbles in the jar are yellow, how many yellow marbles does he have?

Solution: if Richard has x yellow marbles, then the jar has a total of $15 + 20 + x = 35 + x$ marbles. So we know that $\frac{x}{35+x} = \frac{2}{7}$. Cross multiplying gives $7x = 70 + 2x$, solving it we can move all the variables to one side giving $5x = 70$, we divide both sides by 5 to get $x = 14$ so Richard has $\boxed{14}$ yellow marbles.

2 Problem 2

Question: Richard is washing dishes along with his friends Leon and Catherine. Richard can finish all the dishes in 1 hour by him self, Catherine can finish the dishes in 45 minutes by herself, and Leon can finish the dishes in 90 minutes by himself. There is a small problem though, Catherine and Leon like to talk while they work which results in them working 50% slower. How many more hours will it take for Richard to do the dishes by himself compared to all three friends working together? Express your answer as a common fraction.

Solution: Richard finishes $\frac{1}{60}$ of the task every minute, Catherine finishes $\frac{1}{45}$ of the task per minute, but with Leon, she finishes $\frac{1}{90}$ of the task per minute. Leon finishes $\frac{1}{90}$ of the task per minute usually, but with Catherine, he does $\frac{1}{180}$ of the task per minute. We can add these fractions to get how much of the task the three friends complete per minute working together. $\frac{1}{60} + \frac{1}{90} + \frac{1}{180} = \frac{6}{180} = \frac{1}{30}$. Knowing that the three friends complete $\frac{1}{30}$ of the task per minute, it will take them 30 minutes to complete the task. To answer the question, we take the difference between Richards time and the time of all three friends. We can do 60 minutes$-$30 minutes = 30 minutes. The problem asks for the amount of hours as a common fraction, so the answer is $\frac{30}{60} = \boxed{\frac{1}{2}}$

3 Problem 3

Question: 2 Aons are worth 30 Beons and 14 Beons are worth 21 Ceons and each Ceon trades for $\frac{1}{5}$ a Deon. Richard has 10 Aons, how many Deons can he exchange his Aons for?

Solution: 10 Aons = 150 Beons = 225 Ceons = $\boxed{45}$ Deons.

4 Problem 4

Question: What is the area of a triangle with sides $\sqrt{41}$, $\sqrt{53}$, and $\sqrt{58}$? Express your answer as a common fraction.

Solution: Herons formula looks very messy, so we try to find another solution. It seems suspicious that these three side lengths are all square roots which can come from the Pythagorean Theorem. We can notice that $41 = 4^2 + 5^2$, $53 = 7^2 + 2^2$, and $58 = 7^2 + 3^2$. With this, we can realize that we can inscribe the triangle in a 7-by-7 square just like shown in our diagram, Fig. 1. We know that the total area of the square is 49 and the triangles have areas $[MBC] = \frac{21}{2}, [AMN] = \frac{20}{2}$, and $[NDC] = \frac{14}{2}$ adding up to $\frac{55}{2}$. $49 - \frac{55}{2} = \boxed{\dfrac{43}{2}}$

5 Problem 5

Question: Richard is hosting a party for all his math friends. He doesn't know how many friends he invited, but he does know some clues.

- He handed out between 11 and 14 (inclusive) invites on Monday
- He only made 50 invite invite cards
- The number of invite cards he gave out on Tuesday was a perfect square
- The number of invites he gave out on Wednesday was a perfect cube

How many possible ways could Richard have handed out his invites? (14 on Monday, 1 on Tuesday, and 8 on Wednesday is different from 11 on Monday, 4 on Tuesday, and 8 on Wednesday)

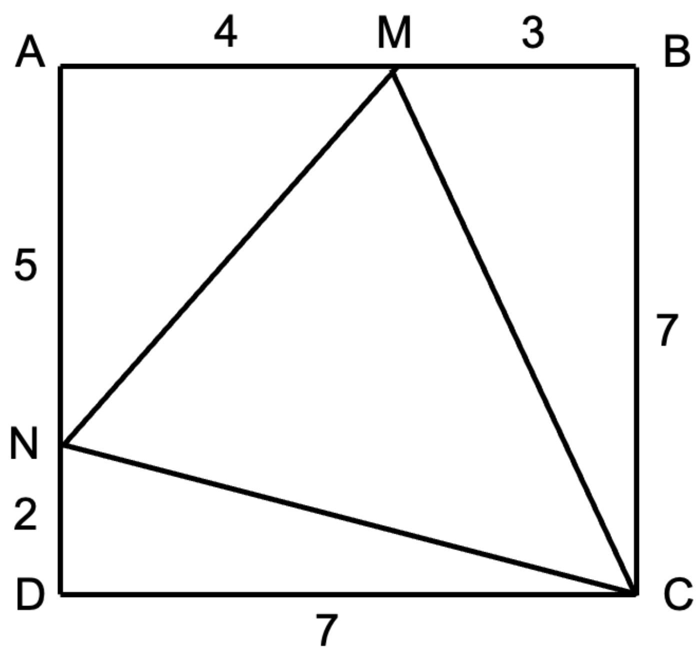

Figure 1: Problem 4 Diagram

Solution: We know Richard invited a maximum of 50 people and invited a total number between 11 and 14 plus a perfect square plus a perfect cube. We can list out the perfect squares and perfect cubes under 39 to find out which combinations have less than 50 total letters. Squares are $1, 4, 9, 16, 25, 36$, cubes are $1, 8, 27$. Because there are less cubes, we can run through all three cases of each cube.

If we choose 27, then when we choose $1, 4$, or 9, they each yield 4 solutions because we get a range of 4 numbers for each one, for example for 9, we get $9+27+11 = 47$ to $9+27+14 = 50$ and 16 is too big because $27+16+11 > 50$. So we got a total of 12 solutions.

If we chose 8, we can use the same method. $1, 4, 9, 16$ or 25 each give 4 solutions, and 36 is too big, so we got 19 more solutions.

If we choose 1, we have to look through $1, 4, 9, 16$, or 25 they give 4 solutions each, but we have to be careful with 36, with it, we get a range of 48 to 51 so it only yields 3 cases. So we got a total of 23 cases

Finally, we add $12 + 19 + 23 = \boxed{54}$ cases

6 Problem 6

Question: Richard is planning to meet up with his friend, Leon. They plan to meet up at Diophantus's diner. They scheduled to meet between 3 o'clock and 4 o'clock. Richard is going to arrive at a random time during this period and wait for 30 minutes. If Leon doesn't arrive, he will leave. Leon forgot he was going to be going out to see a movie right before, so he will arrive at a random time between $3:30$ and $4:00$ and will wait 10 minutes. What is the probability that Richard and Leon do not meet up?

Solution: We should model this with geometric probability because there are an infinite number of cases. We will reference our diagram, Fig. 2, the x-axis is Richard's arrival time and the y-axis is Leon's arrival time. If Richard arrives at $3:00$, Leon must arrive at $3:30$, and the borderline times (times where they have just enough time to meet(this line bounds the area where they can meet)) both increase along the line $x=y$ from that point because if Richard arrives m minute later, Leon can also, so this line goes until it hits the point $(3:30, 3:30)$. From then on, if Richard arrives from $3:30$ $3:40$, he will always meet Leon because When ever Leon arrives, Richard will always already be there, or will be there within 10 minutes. If Richard arrives m minutes after $3:40$, Leon can also, so we draw another line $x=y$ from that point to encompass the area where they meet up (I Richard arrives at $4:00$, Leon needs to arrive between $3:50$ and $4:00$ so the we draw the full corner to connect the line and the side. The total area of the arrival time is 1800 minutes squared and the probability they meet up successfully meet up is 1150 minutes squared. We can simplify the fraction giving $\frac{1150}{1800} = \frac{115}{180} = \frac{23}{36}$. The problem asks for the probability that they don't meet, so we take $1 - \frac{23}{36} = \boxed{\frac{13}{36}}$.

7 Problem 7

Question: What is the minimum value of the expression, $\dfrac{x^4+4x^3+2x^2-4x-3}{x^2-1}$?

Solution: We need to find a way to factor the top before we can think about minimizing. The top looks very symmetrical, suggesting that we can factor it into a polynomial multiplied by another and one of these factor should also be symmetrical. The factorization must be a quadratic times another quadratic, or a linear multiplied by a cubic. We will try a linear

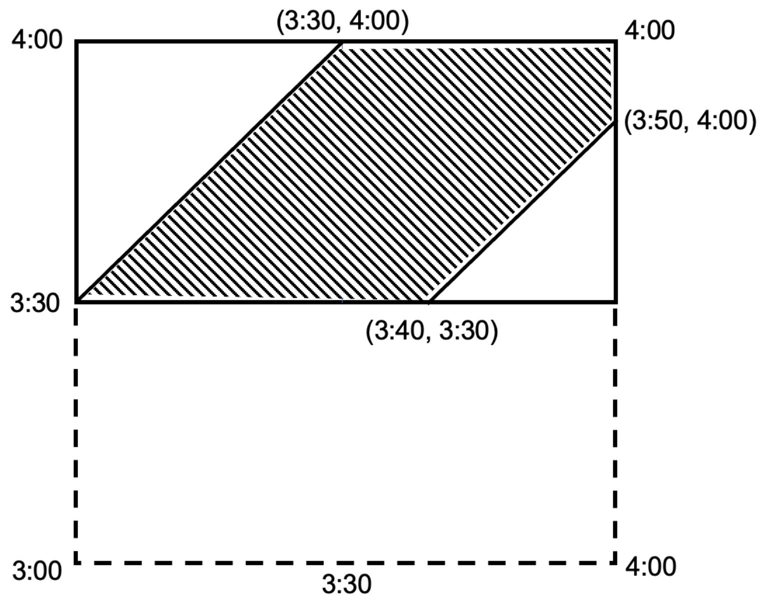

Figure 2: Problem 6 Diagram

multiplied by a cubic first (quadratic by quadratic also works because we will factor down to the same things). To keep it symmetrical, the linear must be $(x+1)$, so when me divide, we get x^3+3x^2-x-3, so we can make the numerator equal to $(x^4+4x^3+2x^2-4x-3) = (x^3+3x^2-x-3)(x+1)$. We see a pattern in the cubic where the cube and quadratic coefficients are the same as the linear and constant allowing us to factor the numerator into $(x^2-1)(x+3)(x+1)$. We know x^2-1 factors into $(x+1)(x-1)$, so now we plug it back in to our original equation to get $\frac{(x+1)(x-1)(x+3)(x+1)}{(x+1)(x-1)}$. We can then simplify, $\frac{(x+1)(x-1)(x+3)(x+1)}{(x+1)(x-1)} = (x^2+4x+3)$. To minimize, we factor x^2+4x+3 into vertex form to get $(x+2)^2-1$. So, the minimum is $\boxed{-1}$. As a side note, you could also guess that x^2-1 was a factor and immediately divided.

8 Problem 8

Question: Richard is going shopping at the Math-O-Mart. He is a premium member, so upon checkout, he is entitled to three coupons to apply to his order. Today, Richard spent \$10 on his order. Math-O-Mart offers him 1, 1-star coupon, a choice of 1 of 2, 2-star coupons, and a choice of 1 of 3, 3-star coupons. Coupons at Math-O-Mart are all functions where x is the price before applying the coupon and $f(x)$ is the price after. He is offered the following 1-star coupon

- $f(x) = x - 10$

He is offered the following 2-star coupons

- $f(x) = x^2 + 2x + 4$
- $f(x) = x^4 + x + 8$

He is offered the following 3-star coupons

- $f(x) = sqrt(x)$
- $f(x) = \frac{1}{x}$
- $f(x) = \frac{x}{10}$

What is the minimal amount Richard will pay for his order, given he can use however many of the coupons he gets and can choose whichever coupons to take?

Solution: We can first focus our attention on the three star coupons because they can all lower our amount by a lot. While taking a square root and dividing by 10 can lower it by a lot, taking the reciprocal of a number greater than one makes it less than 1 so we should create the largest possible number and then take the reciprocal. For the 2-star coupons, the quartic grows much faster than the quadratic, so we should pick that one, both equations only give positive numbers, so we shouldn't worry about using them to get a big negative number. Lastly, we should apply the 1 star coupon last to make our small number negative. Using this method, we put \$10 into $f(x) = x^4 + x + 8$ to get 10018 then we put it into $f(x) = \frac{1}{x}$ to get $\frac{1}{10018}$ then we put it into $f(x) = x - 10$ to get $\boxed{\$ - \frac{100179}{10018}}$