Coordinates and Transformations

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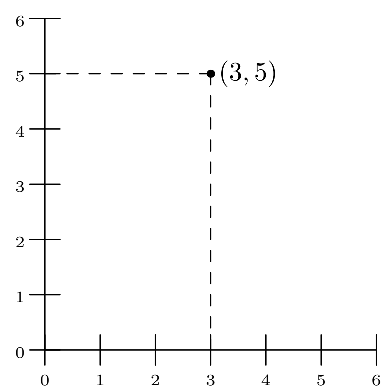
1 Introduction

Our past geometry handouts have focused on *synthetic* methods. Although powerful and elegant, some problems are not easily solvable with such approaches. In these cases, we can often turn to *coordinate* techniques. Coordinates provide a simple and methodical approach, and can work very well for certain problems. However, in other cases they result in very complicated expressions that are not easy to work with.

2 The Cartesian Plane

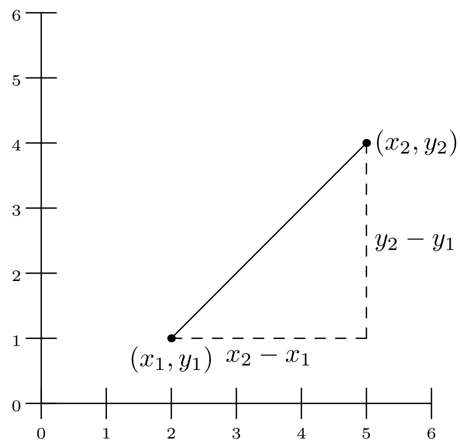
The Cartesian Plane is a grid defined by two perpendicular lines, our x and y axes, where every point on the plane can be represented by a unique ordered pair (x, y), which are defined by the point's distance from the x and y axes. For example, to get to the point $(3, 5)$, we'd start at the origin, $(0, 0)$ and then from there we would move 3 units to the right and then 5 units up. Alternatively, we could move 5 units up and then 3 units to the right, which would still get us to the same point.



2.1 Distance

Let's start by considering two points on the x axis, $(a, 0)$ and $(b, 0)$. From our understanding of the number line, the distance between the two points is $|b - a|$. Similarly, the distance between $(0, c)$ and $(0, d)$ is $|d - c|$. But what if the points aren't on the same axes?

Take two points (x_1, y_1) and (x_2, y_2). We assume that $x_2 > x_1$ and $y_2 > y_1$.



This is a right triangle! Thus by using the Pythagorean theorem, the distance between (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

So now we know how to find lengths in the coordinate plane.

What about ratios? Let's start by considering how to find midpoints.

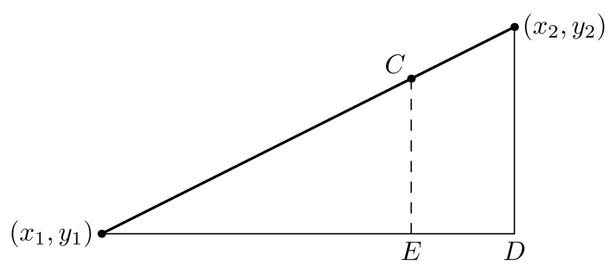
Example 1. Given a line segment with endpoints (x_1, y_1) and (x_2, y_2), what point is the midpoint?

Solution. Let's consider the x coordinate of the midpoint. Clearly it must be precisely between x_1 and x_2, at $\frac{x_1+x_2}{2}$. By similar logic we can find the y coordinate. Thus, the midpoint lies at $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. □

We can generalize this to any division of a line segment into a ratio.

Theorem 2 (Ratio Point Theorem)**.** Let point A be (x_1, y_1) and point B be (x_2, y_2). Point C lies on line segment AB, and $\frac{AC}{CB} = r$. The coordinates of point C are

$$\left(\frac{x_1 + rx_2}{1 + r}, \frac{y_1 + ry_2}{1 + r}\right).$$



Proof. It's clear that $\triangle ACE$ is similar to $\triangle ABD$. Thus, $\frac{AE}{ED} = \frac{AC}{CB} = r$. We can derive AE's length as

$$x_1 + (x_2 - x_1) \cdot \frac{r}{1+r} = \frac{x_1 + rx_2}{1+r}.$$

Thus $E = \left(\frac{x_1+rx_2}{1+r}, 0\right)$. Since C is connected by a vertical to E, it has the same x coordinate. Now, we derive the length of CE, knowing that $\frac{CE}{BD} = \frac{r}{1+r}$. From this we get $CE = (y_2 - y_1) \cdot \frac{r}{1+r}$. The y coordinate of C is thus $\frac{x_1+rx_2}{1+r}$. Finally, we arrive at

$$C = \left(\frac{x_1 + rx_2}{1+r}, \frac{y_1 + ry_2}{1+r}\right).$$

□

With these tools, we can tackle some problems.

Example 3. Let $ABCD$ be a rectangle with $AB = 1$ and $BC = 2$. In addition, let E be a point on diagonal BD such that E trisects the diagonal and is closer to B than D. Find the length of line segment AE.

Solution. Let A be the origin. $B = (0, 1)$ and $D = (2, 0)$, so using the Ratio Point Theorem, we get $E = \left(\frac{2}{3}, \frac{2}{3}\right)$. BE is thus $\frac{2\sqrt{2}}{3}$. □

Let's look at a more advanced problem.

Example 4 (2020 AIME I)**.** A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Walkthrough 1. Coordinates are a good choice for this problem. Let's have O be the origin; when we eventually solve for the coordinates of P, finding OP^2 will be straightforward.

(a) The bug first walks 5 units east. How many times will it turn before it walks to the east again? How many units will it go?

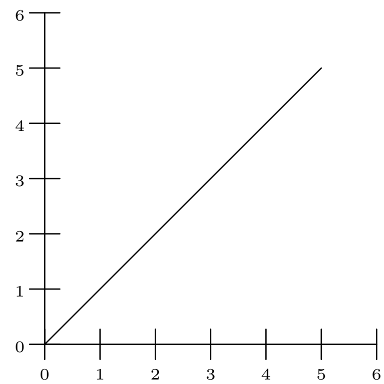
(b) You can find the total distance the bug walks to the east by summing an infinite series.

(c) There are six possible directions for the bug to walk in. How far does it go for each?

(d) Some of the distances will oppose each other, so after subtracting, the bug will move x units to the east, y units at a direction of 60° from east, and z units at a direction of 120° from east.

2.2 Lines

Proposition 5. A *line* is a set of points on the plane that satisfy a linear relation. For example, here's what we get if we plot all points (x, y) for which $x = y$:



As a shorthand, we refer to lines by their relation. So in our example above, we call the graph the "line $y = x$".

Proposition 6. Given two points, we can define a line that passes through both of them.

Suppose we have two points (x_1, y_1) and (x_2, y_2). Then the line

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

will pass through both points, as you should verify. Note that x and y refer to arbitrary coordinates on the line.

Proposition 7. The *slope* of a line is defined as follows. Given two points (x_1, y_1) and (x_2, y_2) on the line, the slope m is equal to

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Proposition 8. The *x-intercept* and *y-intercept* of a line are the two coordinates where it intersects the x axis and y axis, respectively.

Suppose we know the intercepts of a line to be $(b, 0)$ and $(0, a)$. Then we can derive the equation of the line to be

$$y - a = -\frac{a}{b} \cdot (x - b).$$

We rearrange this to

$$ax + by = ab.$$

Proposition 9. Three common forms of linear equations:

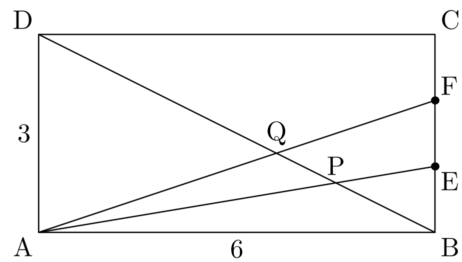
(a) $y_2 - y_1 = m(x_2 - x_1)$.

(b) $ax + by = c$.

(c) $y = mx + b$.

With these tools, we can transplant geometry problems onto the coordinate plane. There, we can use algebra to solve for coordinates of points that we want.

Example 10 (2016 AMC 10A)**.** In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$, where the greatest common factor of r, s and t is 1. What is $r + s + t$?



Solution. Let A be the origin. Then line AF is $y = \frac{1}{3}x$, and line AE is $y = \frac{1}{6}x$. Line BD is $y = 3 - \frac{1}{2}x$, so we solve for the coordinates of Q and P. For the intersection of BD and AF, we have

$$3 - \frac{1}{2}x = \frac{1}{3}x \to x = \frac{18}{5}.$$

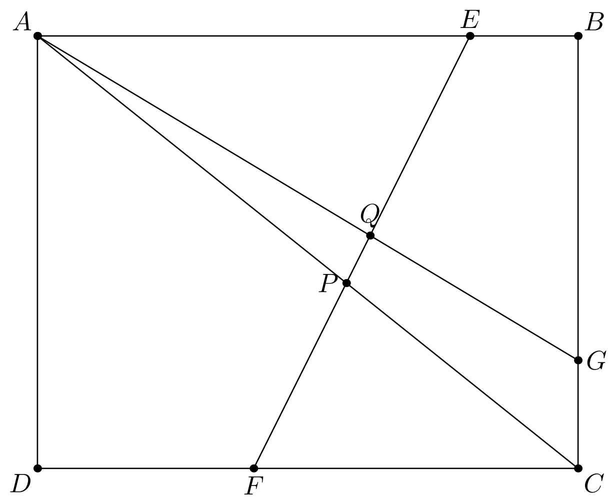
We can immediately derive that $y = \frac{6}{5}$, so $Q = \left(\frac{18}{5}, \frac{6}{5}\right)$. Through a similar process, we find $P = \left(\frac{9}{2}, \frac{3}{4}\right)$. Now, we could use the distance formula to find BP, PQ and QD explicitly, but there's actually no need.

$$\frac{DQ}{DB} = \frac{\frac{18}{5}}{AB},$$
$$\frac{QP}{DB} = \frac{\frac{9}{2} - \frac{18}{5}}{AB},$$
$$\frac{PB}{DB} = \frac{AB - \frac{9}{2}}{AB}.$$

We find $r : s : t$ to be $12 : 3 : 5$, and our answer is $\boxed{20.}$ □

Example 11 (2016 AMC 10B)**.** Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$. and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$?

Solution. Let A be the origin. AG is $y = -\frac{3}{5}x$, and AC is $y = -\frac{4}{5}x$. For line FE, note that the vertical difference is 4 units, and the horizontal difference is 2 units, so the slope is 2. FE passes through $E = (4, 0)$, so its equation is $y = 2x - 8$. We can now solve for Q and P, and get



$Q = \left(\frac{40}{13}, -\frac{24}{13}\right)$ and $P = \left(\frac{40}{14}, -\frac{32}{14}\right)$. There's no need to solve for the length of QP, as $\frac{PQ}{FE}$ equals the horizontal distance between P and Q divided by the horizontal distance between F and E. This is

$$\frac{40}{2} \cdot \left(\frac{1}{13} - \frac{1}{14}\right) = \boxed{\frac{10}{91}}.$$

□

Remark. In the previous two examples, note what point we've chosen to use as the origin. When using coordinate geometry, choice of origin can greatly simplify the algebra required. Although it depends by problem, points that are the intersection of many lines are good candidates.

We will introduce one final tool: perpendicularity.

Proposition 12. If two lines are perpendicular, the product of their slopes is -1.

You can prove this with a similarity argument.

Proposition 13. Take point (x_1, y_1) on line $ax + by = c$. The equation of the perpendicular line that passes through this point is

$$-bx + ay = bx_1 - ay_1.$$

Note that we can derive the line just by knowing the coordinates of the intersection and the slope.

Proposition 14. Take point $P = (x_1, y_1)$ and line $\ell : ax + by + c = 0$. The shortest distance between P and ℓ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

The algebra to prove this is quite involved, so we will omit it.

Example 15 (2019 AMC 10B)**.** Points $A(6, 13)$ and $B(12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

Walkthrough 2. This problem requires a fair amount of knowledge about circle geometry. In fact, synthetic methods often help simplify coordinate approaches.

(a) The circle's center O is equidistant from A and B. What does this imply?

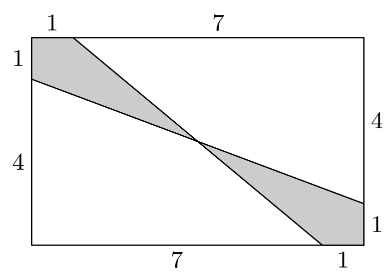
(b) The point at which the tangents intersect is on the perpendicular bisector of AB.

(c) Let M be the midpoint of AB, and X be the intersection of the tangents. Can you see any way to use similarity with right triangles to derive AO?

2.3 Area

Many of our approaches to area are already built around perpendicularity, like $\frac{bh}{2}$ and other formulas. This means that coordinates are well suited for certain problems.

Example 16. What is the area of the shaded region of the given 8×5 rectangle?



Solution. Designate the intersection point as the origin; note that it is also the center of the rectangle. Now we draw the diagonal from the lower left to upper right. This splits the white region into four triangles, which we can quickly compute the total area as $17.5 + 16$. We then find the shaded area by subtracting this from 40, getting $\boxed{6.5.}$ □