Angle Chasing

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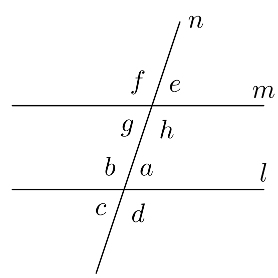
1 Introduction

Angle chasing is one of the simplest-defined but most powerful and most used techniques in geometry. By finding congruent angles and parallel and perpendicular lines, we are able to unlock the world of cyclic quadrilaterals, similar triangles, and many more. Before we begin this handout, proceed knowing that angle chasing can vary from extremely simple to extremely hard - meaning if you get stuck on a problem from this handout, do not be intimidated, and instead continue trying various methods of angle chasing. For those of you who have read AoPS Volume 1, the first few sections of this handout should look familiar to you.

1.1 Parallel Lines



In the figure, we have $l \parallel m$. When we have two parallel lines, such as l and m with a line passing through both of them, such as n, several properties arise.

1. We have the angle equalities $e = g$, $f = h$, $b = d$, and $a = d$. These equalities state that opposing angles in an intersection are equal, and are relatively intuitive.
2. We also have that $a = e$, $b = f$, $c = g$, and $d = h$, or the "corresponding angle" theorems, which state that corresponding angles between the two intersections are equal.
3. Relatively inuitive, because we have many lines, notice that we will also have that $a + d = e + h = a + b = e + f = 180$ by supplementary angles. From this, we can also prove and derive the opposing angle theorems.

1.2 Exercises

Exercise 1.1.1 In triangle ABC, draw a line through A parallel to BC. Using properties of parallel lines, prove the sum of the angles in a triangle is always 180 degrees.

Exercise 1.1.2 In triangle ABC, prove that the exterior angle of A is the sum of the measures of angles B and C. Using this, prove that the sum of the exterior angles is always 360 degrees.

1.3 Similar and Congruent Triangles

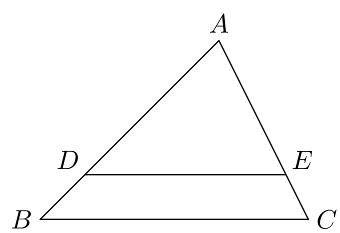
In geometry, angle chasing is often used to find similar triangles, which can simplify your computations greatly by giving way to nice ratios or other nice angles. There are generally three ways to prove similarity in two different triangles.

Given triangles $\triangle ABC$, and $\triangle DEF$, we can prove similarity iff

1. Angle-Angle (AA) Similarity - If triangles $\triangle ABC$ and $\triangle DEF$ are such that $\angle A = \angle D$ and $\angle B = \angle E$, then we have that $\triangle ABC \simeq \triangle DEF$, where $\simeq$ denotes similarity.
2. Side-Angle-Side (SAS) Similarity - If triangles $\triangle ABC$ and $\triangle DEF$ are such that $\angle A = \angle D$ and $\frac{AB}{AC} = \frac{DE}{DF}$, then we have that $\triangle ABC \simeq \triangle DEF$.
3. Side-Side-Side (SSS) Similarity - If we have triangles $\triangle ABC$ and $\triangle DEF$ such that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ then we have that $\triangle ABC \simeq \triangle DEF$.

Note that when we write out if two triangles are similar, the order in which we write the vertices matters - corresponding angles/vertices must be written in the same position in notation.

One common configuration of similar triangles that you may have seen below is where in triangle $\triangle ABC$, points D and E are on sides AB and AC, respectively, so that $BC \parallel DE$.

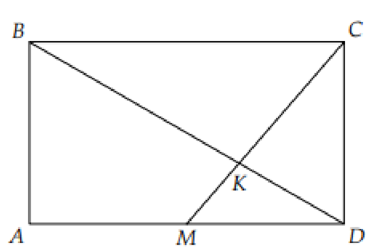


Note here that from our parallel line properties, we have that $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$, so $\triangle ADE \simeq \triangle ABC$. From here, we can find that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}$, which will be useful in many later problems.

1.4 Examples

1.4.1 (May 2015 MATHCOUNTS Mini)

In rectangle $ABCD$, $AB = 6$ units, and the measure of $\angle DBC$ is 30 degrees. M is the midpoint of segment $\overline{AD}$, and segments $\overline{BD}$ and $\overline{CM}$ intersect at K. What is the length of segment $\overline{MK}$?



Solution.
Notice that since BC and MD are parallel, we have that $\angle BDM = \angle DBC$, and by vertical angles, we have that $\angle MKD = \angle BKC$, so by AA-similarity, we have that $\triangle MKD \sim \triangle CKB$.

From here, notice that since M is the midpoint of AD, we have that $\frac{MD}{BC} = \frac{MK}{CK} = \frac{1}{2}$ by similarity, giving us that MK is $\frac{1}{3}$ of MC, or $\sqrt{7}$. (You can find this using the Pythagorean Theorem)

1.5 Exercises

Exercise 1.3.1 If D and E are on sides AB and AC, respectively, of $\triangle ABC$ such that D is the midpoint of AB and E is the midpoint of AC, if $DE = 6$, find BC.

Exercise 1.3.2 If the altitude of a right triangle from the right angle divides the hypotenuse into lengths 4 and 8, find the lengths of the legs of the triangle.

CHALLENGE Exercise 1.3.3 Using similar triangles, prove the Angle Bisector Theorem, which states that in a triangle ABC, if AX is the angle bisector of $\angle BAC$ such that X is on BC, we have that $\frac{BX}{CX} = \frac{AB}{AC}$.

1.6 Hints

1.3.2. Notice that one of the smaller right triangles the altitude splits the bigger right triangle into is similar to the bigger one. What other similar triangles can you find?

1.3.3. If you extend AX to E so that $\angle BEX = \angle BAX$, what similar triangles can you find?

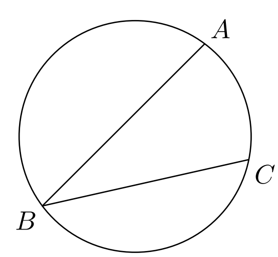
2 Circles

In geometry, you will often encounter circles. In circles, we find that there are many convenient angles that we can "catch" using basic angle identities.

2.1 Basic Circle Identities

2.1.1 Inscribed Angle Theorem

The inscribed angle theorem states that in a circle with points A, B, and C on the circle, the measure of $\angle ABC$ is one half the measure of arc $\widehat{AC}$ not containing B.



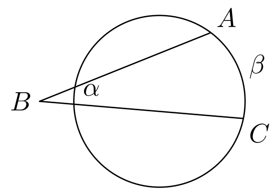
In mathematical terms, we can express this as

$$\angle ABC = \frac{1}{2}(\widehat{AC}).$$

In particular, one should observe that this also implies that angles that intercept the same arc have the same measure.

2.1.2 Exterior Secant Intersection

This theorem states that the measure of an angle formed by two secants of a circle that intersect outside of the circle is equivalent to one half the absolute difference between the measures of the two arcs of the circle the secants intercept.

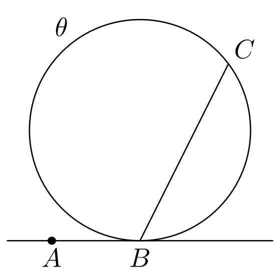


In mathematical terms, we can express this as

$$\angle ABC = \frac{\beta - \alpha}{2}.$$

2.1.3 Intersection of a Tangent and a Chord

The measure of the angle formed by a tangent and a chord of the circle through the tangency point is one half of the measure of the arc that the chord cuts off opposite to the angle.



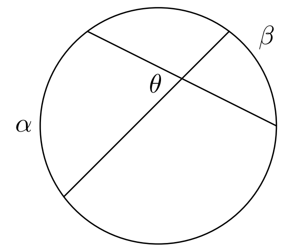
In mathematical terms, this is

$$\angle ABC = \frac{\theta}{2}.$$

2.1.4 Interior Chord Intersections

The measure of an angle formed by two chords that intersect inside the circle is one half of the sum of the two arcs intercepted by the chords.
In mathematical terms, this is equivalent to

$$\theta = \frac{\alpha + \beta}{2}.$$



2.2 Exercises

Exercise 2.1.1 (Thales' Theorem) Prove that given a circle with diameter AB, for any point C on the circle, we have that $\angle ACB = 90$ degrees.

Exercise 2.1.2 (Cyclic Quadrilaterals) For any cyclic quadrilateral $ABCD$ (a quadrilateral that can be inscribed in a circle), prove that

- Opposing angles sum to 180 degrees.
- $\angle ADB = \angle ACB$.

Exercise 2.1.3 (1971 AHSME) Points A, B, Q, D, and C lie on the circle as shown and the measures of arcs $\widehat{BQ}$ and $\widehat{QD}$ are 42 and 38 degrees, respectively. What is the sum of angles P and Q?

2.3 Sidenote

We will not go over the proofs of the identities today, however, if you are curious, they can be found in Chapter 10 of Art of Problem Solving's Volume 1. If you do not own a copy, the proofs can be found at https://drive.google.com/file/d/13PqkhdUkd1rxnTvF2twi7MAguRipouBE/view?usp=sharing.

3 Some More Advanced Example Problems

3.0.1 Canada 1986

Problem.
A chord ST of constant length slides around a semicircle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that $\angle SPM$ is constant for all positions of ST.

Solution.
Let O be the center of the circle. Notice that M, O, P, and S are concyclic, since $\angle SMO + \angle SPO = 90 + 90 = 180$. Therefore, we have that $\angle SPM = \angle SOM$. Since $\angle SOM$ is constant, so is $\angle SPM$, and we are done.

3.0.2 The Orthic Triangle

Problem.
For an acute triangle ABC with orthocenter H, let H_A be the foot of the altitude from A to BC, and define H_B and H_C similarly. Show that H is the incenter of $\triangle H_A H_B H_C$.

Solution.

Notice that H_CHH_AB is cyclic, since $\angle BH_CH + \angle BH_AH = 90 + 90$, or 180 degrees, so we have that $\angle H_CH_AH = \angle H_CBH$. Similarly, we have that H_AHH_BC is cyclic, since $\angle CH_AH + \angle CH_BH = 90 + 90$, or 180 degrees, so we have that $\angle H_BH_AH = \angle H_BCH$.

However, both $\angle H_BCH$ and $\angle H_CBH$ are both equal to $90 - \angle BAC$, so H_AH is the angle bisector of $\angle H_CH_AH_B$.

Proving something similar for the other three angles, we find that H is the incenter of $H_AH_BH_C$, and we are done.

4 Exercises

Note - some of these may not be easy, so do not be discouraged. However, you do know all that you need to know to solve the problems.

4.1 (AoPS Volume 1)

In triangle ADC, a point M is on AC such that $\angle ADM = \angle ACD$. Prove that $(AD)^2 = (AM)(AC)$.

4.2 (EGMO 1.7)

Let O and H denote the circumcenter and orthocenter of $\triangle ABC$, respectively. Prove that $\angle BAH = \angle CAO$.

4.3 (David Altizio)

Triangle AEF is inscribed inside of square $ABCD$ with E on BC and F on CD. If $AE = 4$, $EF = 3$, and $AF = 5$, find the area of square $ABCD$.

4.4 (AHSME 19??)

In triangle ABC, D is in segment BC so that $AC = CD$ and $\angle CAB - \angle ABC = 30$. What is the measure of $\angle BAD$?

4.5 (AMC 2011/10B)

Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. Find the measure of $\angle AMD$.

4.6 (David Altizio's 100 Geometry Problems)

A, B, and C are in a plane such that $\angle ABC = 90$. If D is an arbitrary point on AB, and E is the foot of the perpendicular from D to AC, prove that $\angle DBE = \angle DCE$.

4.7 AIME 2007/II

Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.