Counting & Casework

Catherine Xu

June 16th, 2023

"Counting?"
"I know how to count."

1 Introduction

As we venture into the feared subject of "combinatorics", we ask ourselves a question. What is combinatorics? You may recognize the root of the word, "combi/combo", or combinations, and "ics", which means "the art of", or a profession. In short, combinatorics is really just the art of combinations, and the first step to understanding combinations is learning how to count.

2 Permutations

In our first step towards understanding counting, we attempt to understand permutations. Consider the following problem -

2.1 Opening Problem

How many ways are there to arrange the letters A, B, and C?

Not knowing permutations, you list out the possibilities. Writing them out, you find all of the possibilities are ABC, ACB, BAC, BCA, CAB, and CBA, and you tell me there are 6 possibilities. But that was simple, there were less than 10 possibilities. What if the problem was instead with the letters A, B, C, D, E, F, G, H, I, and J?

2.2 Factorials

You can begin listing out the possibilities to solve that problem, if you like. You'll probably be here for a while, though. So how do we solve this problem? Well, we begin by noting that the first spot has 10 possibilities, and after we've chosen the first spot, there are 9 possibilities left, and so on and so forth, until we have that there are

$$10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1$$

or 3628800 possibilities. You'd probably be here for more than a while if you tried to list all of them.

Such a product, or the product of all the numbers from 1 to n, inclusive, is called a "factorial", or $n!$. Factorials are often used in permutations in combinatorics to better express the equation without having to write out absurdly large numbers, such as 87178291200, or 14!. Interestingly, factorials also arise in many Mathcounts and AMC algebra problems, such as computing the value of

$$\frac{(2013!)(2012!)}{(2010!)(2014!)}$$

which we will cover in a later handout.

2.3 A Generalization

Noticing how we solved the factorial problem, we can also note that if we have k space and each space has a_k possible values, we have that

$$\text{the number of possible sequences} = a_1 * a_2 * \ldots * a_k$$

which we will find to be very useful in later problems.

3 Constructive Counting

3.1 A Note on "Good" Casework

In counting problems, having good casework is very important. When you do casework, you must keep your scratchwork **organized** and be able to later reference cases easily. Otherwise, you will find yourself unable to check for mistakes, and will therefore silly or miss a case easily. Additionally, in a problem, if you are not sure whether order matters or not, you should always assume that it does. This will not make much sense now, but will later. In general, try to keep the number of assumptions you make to close to zero.

3.2 Casework

As the name suggests, casework in counting is the act of splitting the larger problem into smaller problems that are easier to count. In counting, knowing how and when to split the problem into smaller, more manageable cases is a very important and useful skill. Sometimes, the key to a problem itself is knowing what cases to split the problem into.

3.3 Walkthroughs

3.3.1 February 2017 Mathcounts Mini

How many collections of positive, odd integers have a sum of 18? Note that $1+1+1+3+3+9$ and $9+1+3+1+3+1$ are considered to be the same collection.

(a) What's the largest possible positive integer that could be in such a sum?

(b) What's the smallest possible value of the largest positive integer in such a sum?

(c) Take cases by the largest integer in the set.

(d) Once, you've established the largest integer in the set, subtract that off of the 18 and continue to do casework. Do this for all possibilities of the largest number.

3.3.2 CEMC 2012, Gauss 8

Stones are numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Three groups of stones can be selected so that the sum of each group is 11. For example, one arrangement is $1, 10$, $2, 3, 6$, $4, 7$. Including the example, how many arrangements are possible?

(a) Take cases by whichever number is the largest number chosen.

(b) (Largest number is 10) If 10 is used in one of these sets, what number must it be paired with? Use that, and then take further cases by the largest number used.

(c) (Largest number is 9) If 9 is used in one of these sets, what number must it be paired with? Again, use this, and take further cases.

(d) (Largest number is 8) What are the possibilities for the numbers in the same set as 8? Using the x number of cases (where I am not telling you x), again split into further cases.

(e) The largest number in any of these sets cannot be 7 or lower. Can you see why?

4 Complementary Counting

Some days, we find it easier to count things that are **not** it than the things that **are** it. Complementary counting is the act of counting everything that **doesn't** fit the desired description and subtracting it from the total number.

4.1 A Warning

As a warning, one of the most common sillies in complementary counting is **forgetting to subtract the complement from the total**. More often than not, the desired answer and its complement will not be equal, leading you to get the wrong answer despite maybe having the correct logic.

4.2 When to Complementary Count

Complementary counting can be very useful, and knowing when to complementary count, like knowing how to split casework, is a very useful skill in combinatorics. You should complementary count whenever the complement seems much easier to count than the original, whether it be easier or less casework, or simply easier equations. As an example, we begin with a classic heads and tails coin-flipping problem -

4.2.1 Coin Flips

How many ways are there to arrange 7 coins in a row, where you can choose heads or tails, such that there is at least one head?

4.2.2 Coin Flips - Solution

We could take this in casework by the number of coins that are heads and use combinations (which we will cover later), but that would be rather tiring. Instead, we will complementary count the number of arrangements where there are no heads. Noting that there is only one way, when all coins are showing tails, we have that there are

$$2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7 = 128$$

total arrangements. Subtracting the one arrangement where all coins show heads, we have a total of 127 arrangements with at least one head.

4.3 Walkthroughs

4.3.1 AMC 2006 10A, Modified

How many four-digit numbers have at least one digit that is a 2 but not a digit that is a 3?

(a) Use complementary counting.

(b) How many four-digit numbers do not have a digit that is a 3?

(c) How many four-digit numbers do not have a digit that is either a 2 or a 3?

(d) Subtract.

4.3.2 AoPS, "Complementary Counting"

Sally is drawing seven houses. She has four crayons, but she can only color any house a single color. In how many ways can she color the seven houses if at least one pair of consecutive houses must have the same color?

(a) Sense a theme yet?

(b) How many ways are there to color the houses total?

(c) How many ways are there to color the houses such that no two consecutive houses are the same color? (Hint: Remember the Section 2.3 generalization?)

(d) You know what to do.

5 Symmetry

Symmetry is another very useful tactic in combinatorics, and it is mostly used to narrow down the number of cases taken. For example, if you know a problem is symmetric across 7 different cases, instead of taking 7 identical, grueling cases, you can instead do a single case and then multiplying by 7, which could potentially save you lots of time in-test.

5.1 A Note on "Assuming" Symmetry, Engineer's Induction

Like you should not assume that angle in that geometry problem is right because the figure is not drawn to scale because it could be 89 degrees, you should not make a habit out of assuming symmetry in combinatorics, as it could lead you to getting some very wrong answers. While solving a problem, if you say some x number of cases are symmetric, you must prove that these cases are symmetric.

5.2 Example

5.2.1 AoPS's Intermediate Counting & Probability

In how many different ways can 6 people be seated at a round table? Two seating arrangements are considered the same if, for each person, the person to his or her left is the same in both arrangements.

5.2.2 Solution

Note that originally, we would have exactly 6!, or 720 different arrangements of the people around the table. However, each time we rotate a single table, we get another arrangement that is counted differently but should be counted as the same. Since each table has 6 different rotations, we divide by 6 to get the number of arrangements where rotation does not matter, or 120 total seatings.

5.3 Walkthroughs

5.3.1 Purdue University, "Counting Problems Involving Symmetry"

How many dice can be formed by labeling the sides of a cube with the numbers 1, 2, 3, 4, 5, and 6? Note that if one die can be rotated to form another, then these two die are counted as the same die.

(a) If rotations did matter, how many ways would there be to label the cube?

(b) Fix whichever side is on the bottom. How many ways are there to rotate it now?

(c) How many total rotations are there now, and how can you use that to get rid of the symmetries? (Hint: $\div$)

5.3.2 A Three Year Mathcounts Marathon, Karen Ge

Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.

(a) This is just like the last problem! Think of the colors as the numbers 1-8. How many ways are there to rotate the octahedron?

(b) Establish an "anchoring point" at the bottom of the octahedron - how many possible "anchoring points" are there?

(c) Now that you have an "anchoring point", how many ways are there to rotate the octahedron, with the "anchoring point" still fixed?

(d) How many ways would there be to originally color it, had rotations mattered?

(e) You know what to do now.

6 Overcounting

In taking casework, we might find it harder to take disjoint cases, and may find it easier instead to double count some cases, and subtract those later. This is called overcounting, and can usually be easily fixed through division or subtraction.

6.1 Accounting for Overcounting with Division

Firstly, notice that any symmetry problem is a problem that is overcounting that is corrected through division. Otherwise, we have other problems, such as permutations, that use division as a technique to fix overcounting. For example, we have the following problem below.

6.1.1 AoPS's Intermediate Counting & Probability

How many possible distinct arrangements are there in the word BALL?

6.1.2 Solution

Notice that the answer is **not** just 4!. Now, notice that if we let the L's be distinct from each other, for example, L_1 and L_2, then it **would** be 4!. However, there are two ways to arrange these L's, had they been alone, and they are counted as the same type of L in the problem. Therefore, we notice that both cases are symmetric, and we can just divide our original answer of 4! by 2 to get an answer of 12.

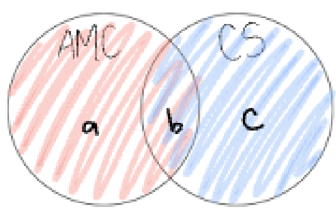
6.2 Double Counting (PIE for $n = 2$)

Often, you will probably come across problems that are referred to as "Venn Diagram" problems, such as the following.

6.2.1 Opening Problem

Iowa City Math Circle offers two courses, one AMC 10 class and one coding session. If 20 students attend the coding session, 14 attend the AMC 10 class, and 12 attend both, how many students attended at least one session?

It might be tempting to add all the numbers together to get an answer of 46, but you would be double counting many students. So how would you solve this? We begin with a Venn Diagram.



Let a be the number of people in only the AMC class, b be the number of people in both, and c be the number of people in only the coding session. Notice that we have that b is 12, $a + b = 14$, and $b + c = 20$. It all becomes clear now. Solving, we find that $a = 2$ and $c = 8$, which gives us that the total number of students is $a + b + c$, or 22.

In general, we have that

$$\text{people in A or B} = \text{people in A} + \text{people in B} - \text{people in A and B}.$$

which will be useful in many problems like these.

6.3 Walkthroughs

6.3.1 ICMC Instructors

In NWJH's band, there are 20 people who play woodwinds, 14 who play percussion, and 8 who play both. If people in NWJH's band only play woodwind and/or percussion instruments, how many people are there in NWJH's band?

(a) Think of the woodwinds as the AMC class and the percussionists as the coding session students! What can you figure out?

(b) In the generalization, A is people in woodwinds, and B is people who play percussion.

6.3.2 ICMC Instructors

Iowa City Math Circle has a committee of 12 handout writers. 5 of these writers write combo handouts, while 10 writers write geometry handouts. If ICMC only writes geometry and combo handouts, how many ICMC handout writers write both geo and combo handouts?

(a) This is not like the typical problem. Try to set what A and B are in this situation.

(b) If a is the number of people only in A, b the number of people only in B, and c the number in both, what can you say about $a+c$, $b+c$, and $a+b+c$?

(c) Using this information, solve for a, b, and c.

(d) Which of these variables does the problem ask for?

7 Combinations

Another foundation of counting is the idea of ”combinations” - also known as the ”choose” function, or the act of choosing k objects from a group of n distinct objects. But is there a simple way to denote this other than just listing out all of the possibilities?

7.1 Simple Committee Problems & Combination Derivations

In this section, we will explore the derivation of the combination formula.

7.1.1 Opening Problem

Consider a club with n people. How many ways are there to form a k-person committee from the total of n people?

We can start by counting the number of ways to choose k people if order matters. There are n choices for the first person, $n-1$ for the second, and so on and so forth, until there are $n-k+1$ choices for the k-th person. Multiplying these together, we have

$$n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

ways to choose the k people **if order matters**. However, since order does not matter, we must divide by $k!$, which gives us the combination formula, or

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

These problems can also arise to other classic problems, such as the following,

7.1.2 Grid Walking

How many ways are there to walk from $(0,0)$ to $(4,3)$ in the plane, if each step must be one unit upwards in the y-direction or one unit right in the x-direction?

Notice that each path is a combination of 4 R's and 3 U's, where R represents a move to the right and U represents a move up. In general, the idea is that out of 7 moves, we want to choose 4 to be to the right. This gives us a total of $\binom{7}{4}$ or 35 possible paths.

7.2 A Note on Combinations

While it is usually not recommended that you memorize things without understanding them, you may find it useful to memorize the first few combinations to speed up your computation processes in the future.

7.3 Stars & Bars, Balls & Boxes, Sticks & Stones

Many of you may have heard of this technique, we begin with an opening problem to understand the applications of this.

7.3.1 Positive Integers

How many ordered triples of positive integers (x, y, z) exist such that $x + y + z = 6$?

Obviously, since 6 is a small number, we can begin listing out all the possibilities. We find that we have $(4,1,1)$, $(1,4,1)$, $(1,1,4)$, $(3,2,1)$, $(3,1,2)$, $(2,1,3)$, $(2,3,1)$, $(1,2,3)$, $(1,3,2)$, and $(2,2,2)$, for a total of 10 possibilities. But is there a better way to do this? We can use a technique called **stars and bars** to finish the problem with a simple combination formula.

Letting a star represent a quantity of 1, we can represent 6 as

$$* * * * * *$$

and using two dividers, we could represent a sum of 6 by using the two dividers to divide 6 into three different quantities. For example,

$$* * *| * *|*$$

represents the ordered triple $(3,2,1)$. Since there are $6-1$, or 5 slots for dividers, and $3-1$, or 2 dividers needed, we have a total of $\binom{6-1}{3-1}$, or 10 triples total.

In general, if we have k positive integers summing up to n, we would have $\binom{n-1}{k-1}$ total k-tuples.

7.3.2 Nonnegative Integers

How many ordered triples of nonnegative integers (x, y, z) exist such that $x + y + z = 6$?

Notice that in this case, it is essentially the same as the previous problem, except dividers can now be placed next to each other. For example, the arrangement

$$* * * * || * *$$

represents the ordered triple $(4,0,2)$. In this case, there are now $6+3-1$ slots to place dividers, and $3-1$ dividers to be placed. Therefore, we have that we have $\binom{6+3-1}{3-1}$, or 36 total ordered triples, which would be quite exhausting to list out.

In general, you may find that if we have k nonnegative integers summing up to n, the total number of ways to do so would be $\binom{n-k+1}{k-1}$, using similar derivations as shown above.

7.3.3 Applications of Stars and Bars in Other Counting Problems

Suppose you have 8 apples and 3 friends, Milly, Nilly, and Pilly. How many ways are there to split all 8 apples among your three friends if you want to give your best friend Milly at least 2 apples?

First, we can give Milly 2 apples, leaving us with 6 apples to split among 3 friends. Notice that if we let the number of apples Milly has not including the 2 you already gave Milly, be m, the number of apple Nilly has be n, and the number of apples Pilly has be p, we have the equation

$$m + n + p = 6$$

which we recognize as our stars and bars equation. Since m, n, and p are nonnegative, we have a total of $\binom{6+3-1}{3-1}$, or 36 ways to split your 8 apples.

7.4 Walkthroughs

7.4.1 ICMC Instructors

For any nonnegative integers n and k such that $k \leq n$, prove that $\binom{n}{k} = \binom{n}{n-k}$.

(a) Write out the formula for both sides of the equation. What do you notice?

(b) Is there a simpler argument that doesn't involve listing out formulas? How is choosing k to be in the group the same as choosing $n - k$?

(c) Notice that choosing k to be in the group is the same as choosing $n - k$ to not be in the group.

7.4.2 AIME 1998/7

Find the number of ordered quadruples of odd integers (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 98$.

(a) Notice that all odd integers can be expressed in the form of $2k - 1$ for some positive integer k. Let x_i be expressed as $2k_i - 1$. What do you notice?

(b) Add the equations together, add 4 to both sides, and divide both sides by 2. What equation are you left with?

(c) You should now have that $k_1 + k_2 + k_3 + k_4 = 51$ for positive integers k_i. Now use Stars and Bars to finish the problem.

8 Exercises

8.1 AMC 10A/2003/21

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

8.2 Mock AIME 2005/2

Find the number of 7 digit positive integers that have the property that their digits are in increasing order.

8.3 Art of Problem Solving

How many six-degree polynomials $f(x)$ with positive integer coefficients are there such that $f(1) = 30$ and $f(-1) = 12$?

8.4 AIME 1984/11

A gardener plants three maple trees, four oaks, and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Find the probability that now two birch trees are next to each other.

8.5 February 2017 Mathcounts Mini

In a 5×5 grid of equally spaced dots, how many different squares of any size can be drawn by connecting four of the dots?

8.6 A Three Year Mathcounts Marathon, Karen Ge

From the first twenty positive integers, how many ways can we select 6 integers so that there are no consecutive integers among the six chosen integers?

8.7 Mathcounts

Two red, two yellow, and two green faces, all unit squares, are available for building a cube. How many distinct cubes can be built?

8.8 Russia

There are 12 books on a shelf. How many ways are there to choose five of them so that no two of the chosen books stand next to each other?

8.9 A Three Year Mathcounts Marathon, Karen Ge

Encode the letter A as 0, B as $1, \ldots, Z$ as 25. How many 5-letter words have the sum of the codes of its letters equal to 14?