Digits and Bases Handout

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1 Introduction

1.1 Digits

Numbers are written as series of *digits*, such as 4, 32, 1735, and so on. We can expand these numbers; for example, $1735 = 1 \cdot 1000 + 7 \cdot 100 + 3 \cdot 10 + 5$. Let's see how this basic decomposition lets us solve some basic math problems.

Example 1.1.1. Prove that if a number's digits sum to a multiple of 9, it is divisible by 9.

Solution: First, any number can be represented as $x_n \cdot 10^n + x_{n-1} \cdot 10^{n-1} + \cdots + x_0$. Let's consider the remainder when we divide this number by 9.

For any number 10^n, the number $10^n - 1$ will be $999 \cdots 999$, where there are n 9s. Obviously, this is a multiple of 9. So when you divide 10^n by 9, the remainder will be 1.

If you multiply 10^n by some number a, the remainder will be $1 \cdot a = a$. This means that when you divide $x_n \cdot 10^n + x_{n-1} \cdot 10^{n-1} + \cdots + x_0$ by 9, the remainder reduces to $x_n + x_{n-1} + \cdots + x_0$! If this sum is a multiple of nine, then we can further reduce the remainder to 0, meaning that we have proven that when a number's sum of digits is a multiple of nine, it itself is also a multiple of 9.

Exercise 1.1.1. Prove the divisibility by 3 rule: if a number's digits sum to a multiple of 3, that number is also an multiple of three. *Hint:* Start with the divisibility by 9 proof; how can you modify it?

Exercise 1.1.2. A particular fortune cookie lists four two-digit positive integers as your lucky numbers. The first three are 57, 13, and 72, but you got sauce on the last one and can't read it. If the sum of the digits of all four numbers equals $\frac{1}{5}$ of the sum of all four numbers, what is the smallest possibility for the fourth lucky number? (Alcumus)

Exercise 1.1.3. A two-digit integer AB equals $\frac{1}{9}$ of the three-digit integer AAB, where A and B represent distinct digits from 1 to 9. What is the smallest possible value of the three-digit integer AAB? (Alcumus)

1.2 Bases

Recall our expansion of numbers. When we decomposed 1735 into $1000 + 700 + 30 + 5$, we were using powers of 10. However, there's no reason why 10 is special; we can use any integer as the power. For example, 57 is equivalent to $2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0$, which we can represent as 212_5. This brings us to the following notation:

Definition 1.2.1. The number $a_n a_{n-1} a_{n-2} \cdots a_0$ in base k is $a_n \cdot k^n + a_{n-1} \cdot k^{n-1} + \cdots + a_1 \cdot k^1 + a_0$.

We can convert from any one base to another.

Example 1.2.1. Convert 57_{10} into base 2.

Solution: When converting between bases, it helps to know the powers for a base. In this case, we know that 32 is 2^5, so we have our first digit: 1. $57 - 32 = 25$, and the next largest power of 2 is 16, so our next digit is also an 1. $25 - 16 = 9$, so we can subtract 2^3 to get 1. We can't subtract 4 or 2, so those place values are left as 0, leaving the ones place as 1. $57_{10} = 111001_2$.

When the original base is not base 10, it becomes more complicated.

Example 1.2.2. Convert 324_6 to base 7.

Solution: There isn't much you can do besides convert 324_6 to base 10, then convert that to base 7.

$$324_6 = 3 \cdot 36 + 2 \cdot 6 + 4 = 108 + 12 + 4 = 124.$$

Now we can separate 124 into powers of 7:

$$124_{10} = 49 \cdot 2 + 7 \cdot 3 + 5 = 235_7.$$

However, in some cases there are clever tricks we can utilize.

Example 1.2.3. Convert 100110101011_2 to base 16.

Solution: Notice that 16 is a power of 2, specifically 2^4. This allows us to partition 100110101011_2 into groups of 4:

$$1001|1010|1011_2$$

We then can evaluate each group separately into $9, 10$, and 11, which using base 16 is $9AB_{16}$.

Exercises

Exercise 1.2.1. What is 441_{10} in base 7?
Exercise 1.2.2. What is the product of the digits in the base 8 representation of 6543_{10}? (Alcumus)
Exercise 1.2.3. What is the greatest 3-digit base 8 positive integer that is divisible by 5? (Express your answer in base 8.) (Alcumus)
Exercise 1.2.4. A certain integer has 4 digits when written in base 8. The same integer has d digits when written in base 2. What is the sum of all possible values of d? (Alcumus)

2 Problem Solving Techniques

Let's take a look at the ways we can solve some advanced problems using these techniques.

2.1. Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{nine}$, where a, b and c are (not necessarily distinct) digits. (2022

AIME I #2)

Solution: We can rewrite the problem statement as

$$100a + 10b + c = 81b + 9c + a.$$

Rearranging this gives $99a = 71b + 8c$. We could use guess and check from here, but a faster solution is gained from noticing that $99 - 71 = 28$. These means that if we let a and b be both equal to 2, we have $56 = 8c$, for $c = 7$. Therefore, our answer is **227**.

The above problem supplied us with bases and asked us to find the digits. However, some problems will ask for the reverse.

2.2. In the equation below, A and B are consecutive positive integers, and A, B, and $> A + B$ represent number bases:

$$132_A + 43_B = 69_{A+B}.$$

What is $A + B$? (2012 AMC12B # 11)

Solution: First, we expand out the equation:

$$A^2 + 3A + 2 + 4B + 3 = 6(A + B) + 9.$$

We know that A and B are consecutive numbers, so we can substitute $A - 1$ and $A + 1$ and solve. For $B = A - 1$:

$$A^2 + 3A + 2 + 4A - 4 + 3 = 6A + 6A - 6 + 9.$$

This simplifies to $A^2 - 5A - 2 = 0$, which has no integer solutions. For $B = A + 1$:

$$A^2 + 3A + 2 + 4A + 4 + 3 = 6A + 6A + 6 + 9.$$

This simplifies to $A^2 - 5A - 6$. The positive solution to this is 6, so we have $A = 6$ and $B = 7$. Their sum is therefore **13**.

Let's take a look at some digit problems.

2.3. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer. (2006 AIME I #3)

Solution: Let's call the number that has its leftmost digit removed x. Its leftmost digit will be a. From the problem, we can write

$$\frac{x}{x + a \cdot 10^n} = \frac{1}{29}.$$

Note that n is the number of digits in x.
Now, we multiply out the denominator to get

$$29x = x + a \cdot 10^n.$$

We rearrange this to $28x = a \cdot 10^n$. We can see that a must be 7, as there is no factor of 7 in 10^n. Next, this means that we must have a factor of 4 in 10^n. The smallest number that satisfies this is 100, so we have $28x = 7 \cdot 100$. We solve to get $x = 25$. This means that the original number is **725**. As a check, notice that $29 \cdot 25 = 725$.

Some problems will ask you to construct a number with digits that satisfy some requirement.

2.4. The integer n is the smallest positive multiple of 15 such that every digit of n is either 8 or 0. Compute $\frac{n}{15}$. (1984 AIME # 2)

Solution: First, note that the number must both be a multiple of 3 and 5. As we proved earlier, if a number's digits sum to a multiple of three, that number is a multiple of three. This means that there will be 3 digits of 8 in the number. In addition, the number must end in 0 to be divisible by 5. This gives us 8880 as the number. We divide by 15 to get **592.**

3 Problems

Stars indicate more difficult problems.

1. A base-10 integer n can be represented as 32_a in one base and 23_b in another base, where a and b are any integer bases larger than 3. What is the smallest possible sum $a + b$? (Alcumus)

2. When 555_{10} is expressed in this base, it has 4 digits, in the form ABAB, where A and B are two different digits. What base is it? (Alcumus)

3. For what integer b is the base-b number $2021_b - 221_b$ not divisible by 3? (2021 AMC 10A # 11)

4. The numeral 47 in base a represents the same number as 74 in base b. Assuming that both bases are positive integers, what is the least possible value of $a + b$? (1971 AHSME #11)

5. A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. What is the middle digit? (1968 AHSME #33)

6. If 554 is the base b representation of the square of the number whose base b representation is 24, then what does b, when written in base 10, equal? (1973 AHSME #6)

7. In the base ten number system the number 526 means $5 \times 10^2 + 2 \times 10 + 6$. In the Land of Mathesis, however, numbers are written in the base r. Jones purchases an automobile there for 440 monetary units (abbreviated m.u). He gives the salesman a 1000 m.u bill, and receives, in change, 340 m.u. What is the base r? (1961 AHSME #17)

8. A base-10 three digit number n is selected at random. What is the probability that the base-9 representation and the base-11 representation of n are both three-digit numerals? (2003 AMC 10A #20)

9. Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N? (2001 AMC12 #2)

10. The product $(8)(888\ldots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k? (2014 AMC 10A #20)

*11. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with the terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime factor that always divides S? (AMC 12A #11)

*12. Hexadecimal (base-16) numbers are written using numeric digits 0 through 9 as well as the letters A through F to represent 10 through 15. Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n? (2015 AMC 10A #18)

*13. Suppose A, R, S, and T all denote distinct digits from 1 to 9. If $\sqrt{STARS} = SAT$, what are A, R, S, and T? (2017 UNM-PNM STATEWIDE HIGH SCHOOL MATHEMATICS CONTEST II #2)

*14.The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (2007 AMC 12B #21)