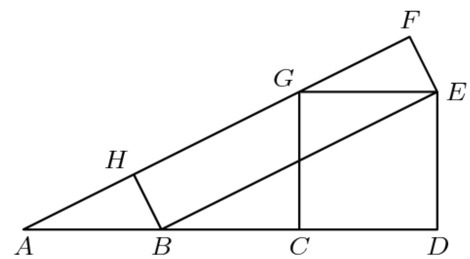
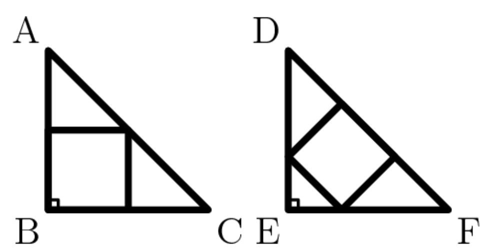
In this handout we'll show how you can use basic algebra to solve geometry problems from math competitions.

1. (MATHCOUNTS State Team 2009) In the figure below, quadrilateral *CDEG* is a square with *CD* = 3, and quadrilateral *BEFH* is a rectangle. If *BE* = 5, how many units is *BH*? Express your answer as a mixed number.



Solution: First, note that *BDE* is a 3-4-5 right triangle. Next, note that *GFE* is similar to *BDE* by angles, so $\frac{FE}{GE} = \frac{BE}{ED}$. Plugging in the values we know, we can derive $FE = \frac{9}{5}$. *FE* = *BH*, so we have $BH = \frac{9}{5}$.

2. (MATHCOUNTS National Target 2005) Triangles ABC and DEF are congruent right isosceles triangles. The square inscribed in ABC has an area of 15 square units. What is the area of the square inscribed in DEF?



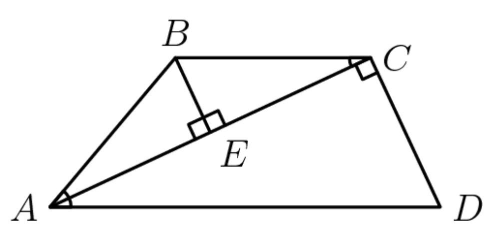
Solution: The square in ABC has a side length of $\frac{1}{2}$ of AB. Let the point where the square meets ED be called P. We can determine that $EP \cdot \sqrt{2} = DP \div \sqrt{2}$ by similar triangles, so DP = 2EP. The side length of the inscribed square is $\sqrt{2}$ times EP, which is $\frac{1}{3}$ of the length of DE. So, the area of the square is $\frac{2}{9} \cdot 4 = \frac{8}{9}$ of the square in ABC. Our answer is therefore $\frac{40}{3}$.

3. Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on AB with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$?

Solution: Call the length of EB x. The length of AB is therefore $8x$. We can find the length of EF using the Pythagorean theorem. Right triangle EBF has legs of side length x and $7x$, so the hypotenuse has length $x\sqrt{50}$. Therefore, the ratio of the area of EFGH to ABCD is $\frac{50x^2}{64x^2} = \frac{25}{32}$.

4. Suppose that $ABCD$ is a trapezoid in which $\overline{AD} || \overline{BC}$. Given $\overline{AC} \perp \overline{CD}$, $\overline{AC}$ bisects angle $\angle BAD$, and $[ABCD] = 42$, then compute $[\triangle ACD]$.

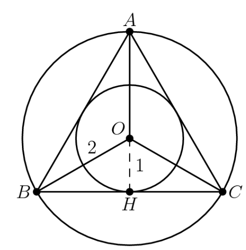
Solution: Our diagram should look like this:



The key to solving this problem is noticing that angle BCA is equal to angle DAC. Since angle BAC is equal to DAC, this means that ABC is isosceles. Furthermore, this means we can draw an angle bisector from B to AC which creates two right triangles that are similar to the triangle ACD. These two triangles have half the side length of ACD, so their combined area is $\frac{1}{4} \cdot 2 = \frac{1}{2}$ of DAC. This means that DAC is $\frac{2}{3}$ of the area of ABCD, so our answer is 28.

5. Two concentric circles have radii 1 and 2. Two points on the outer circle are chosen independently and uniformly at random. What is the probability that the chord joining the two points intersects the inner circle?

Solution: Without loss of generality, let one point on the outer circle be A. As the diagram shows, the arc along BC is the only range where the second point can be and satisfy the problem's condition. This arc is 120 degrees, so the probability is $\frac{1}{3}$.



6. Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?