Distance and Rate

Richard Yang

June 2022

1 Introduction

This handout deals with rates. A rate is essentially how much of something is accomplished in a certain amount of time. For example, when you drive in a car at 40 miles per hour, it means that for every hour you drive, you will have traveled 40 miles, or if a pump can deliver water at 5 liters per minute, it means that every minute 5 liters will have been pumped.

Example 1. A boy walks at a rate of four miles per hour to school, but jogs six miles per hour on the way back. What is his average speed?

Solution: Firstly, the answer is not 5 miles per hour! To see why, let's say the distance the boy walked is x. Then it takes him $\frac{x}{4}$ hours to get to school, and $\frac{x}{6}$ hours to get back. In total, he spends $\frac{x}{4} + \frac{x}{6} = \frac{5x}{12}$ hours. In addition, the amount of distance he covers is $2x$. We divide the distance by the time to find the rate, and the x term cancels, giving us $\frac{24}{5}$ miles per hour.

When we have two numbers a and b, their *harmonic mean* is $\frac{2}{\frac{1}{a}+\frac{1}{b}}$. Solving this type of problem is equivalent to finding the harmonic mean. However, this type of problem is dependent on **the distance traveled being the same.**

Example 2. A girl walks to school at a rate of four miles per hour. When school ends, she decides to go to a nearby store, which is twice as far away as her home. If her walking rate is three miles per hour on this trip, what is her average speed for both trips?

Solution: Once again, we can call the distance from home to school x. The amount of time expended is $\frac{x}{4} + \frac{2x}{3} = \frac{11x}{12}$. The amount of distance covered is $3x$. We divide to get $\frac{36}{11}$ miles per hour.

Note that we used the same technique as the first problem. Often times, knowing a specific tool is not required to solve a problem if you know how to apply the basic concept behind it.

Example 3. A pump can fill a swimming pool in three hours. Another pump can fill it in six hours. If there is a third pump, and when all three are running, it takes one hour to fill in the pool, how many hours does it take the third pump to fill the swimming pool alone?

Solution: Pump A can fill the swimming pool in three hours, so every hour it can fill $\frac{1}{3}$ of the pool. Likewise, Pump B can fill $\frac{1}{6}$ of the pool per hour. Pump C can fill x of the pool in one hour. When we sum Pump A, B, and C's rates, they should equal one. We can solve for x through the equation $1 - \frac{1}{3} - \frac{1}{6} = x = \frac{1}{2}$. Therefore, it would take Pump C two hours to fill the pool alone.

Checkpoints

1. Lorri took a 240 km trip to Waterloo. On her way there, her average speed was 120 km/h. She was stopped for speeding, so on her way home her average speed was 80 km/h. What was her average speed, in km/h, for the entire round-trip? (CEMC 2007 Gauss 8)
2. Brad bicycles from home at an average speed of 9 miles per hour until he gets a flat tire. With no way to fix the tire, Brad walks his bike back home by the same route, averaging 3 miles per hour. If the entire round trip of biking and walking took a total of 6 hours, what was Brad's average speed in miles per hour for the entire round trip? Express your answer as a decimal to the nearest tenth. (MATHCOUNTS)
3. Frank can mow his own yard in 45 minutes. Joe's yard is 40% larger than Frank's, but Joe can mow his own yard in an hour. How long would it take them, working together, to mow both yards? Express your answer as mm:ss, where mm is the number of minutes and ss is the number of seconds, rounded to the nearest whole number, it would take them. (Alcumus)
4. If Eric can paint 3 cars in 4 hours and 2 trucks in 5 hours, then how long, in hours, would it take him to paint 4 cars and a truck? Express your answer

as a common fraction.
5. One night two cylindrical wax candles of different heights and different diameters were lit. One of the candles was 20 cm taller than the other. They were both lit at the same time and each burned at a steady rate. Five hours after they were lit they were both the same height. The taller one burned all of its wax six hours after it was lit, and the shorter one burned all of its wax 10 hours after it was lit. What was the ratio of the original height of the shorter candle to the original height of the taller candle? Express your answer as a common fraction. (MATHCOUNTS)

2 More Advanced Problems

Example 1. A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person. (AIME 2007 I)

Solution: We can set up equations to represent the position of each person at a certain time. Al's position is described by $A = 6t$, where t is the number of seconds since he has stepped on the walkway. Bob's position is $B = 10(t-2)$, and Cy's position is $C = 8(t-4)$. There are three cases, where either Al, Bob or Cy is in the middle. If Al is in the middle, then his position can be descibed by the average of Bob and Cy's position, or $A = \frac{B+C}{2} = 9t-26 = 6t$. We solve the equation to get $t = 26/3$ in this case. If Bob is in the middle, we have $B = \frac{A+C}{2} = 7t-16 = 10t-20$. This gives a negative value of t, which is invalid. Finally, if Cy is in the middle, we have $C = \frac{A+B}{2} = 8t-10 = 8t-32$. This has no solutions. Therefore, the only valid solution is $t = 26/3$. At this time, Al was $6 \cdot \frac{26}{3} = 52$ feet down the walkway.

Example 2. Rowena can paint a room in 14 hours, while Ruby can paint it in 6 hours. If Rowena paints for x hours and Ruby paints for y hours, they will finish half of the painting, while if Rowena paints for y hours and Ruby paints for x hours they will paint the whole room. Find the ordered pair (x, y). (Alcumus)

Solution: We can set up a system of equations:

$$\frac{x}{14} + \frac{y}{6} = \frac{1}{2}.$$

$$\frac{y}{14} + \frac{x}{6} = 1.$$

We simplify denominators to $3x + 7y = 21$ and $7x + 3y = 42$. From here, we use basic algebra techniques to derive $(x, y) = \left(\frac{231}{40}, \frac{21}{40}\right)$.

Example 3. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before? (AIME 2004 II)

Solution: Let's say the project was scheduled to be completed in 4 units of time. The first quarter of the project is completed on time. However, the second quarter is completed with $\frac{9}{10}$th of the workers, and thus takes $\frac{10}{9}$ units of time. Applying the same logic, the third quarter is completed in $\frac{5}{4}$ time units. We now have $4 - (1 + \frac{10}{9} + \frac{5}{4}) = \frac{23}{36}$ time units left to complete the final quarter. In order to finish on time, we need at least $1000 \cdot \frac{36}{23}$ workers, which to the nearest integer is 1566. Therefore, the company needs to hire $1566 - 800 = 766$ workers.

Example 4. Zuleica's mother Wilma picks her up at the train station when she comes home from school, then Wilma drives Zuleica home. They always return home at 5:00 p.m. One day Zuleica left school early and got to the train station an hour early. She then started walking home. Wilma left home at the usual time to pick Zuleica up, and they met along the route between the train station and their house. Wilma picked Zuleica up and then drove home, arriving at 4:48 p.m. For how many minutes had Zuleica been walking before Wilma picked her up? (Alcumus)

Solution: Instead of blindly applying our tools, let's think more carefully. Wilma leaves at the same time, but gets home 12 minutes early. This means that Zuleica was picked up six minutes earlier then usual. In addition, she arrived at the train station 60 minutes early. So, the amount of time she was walking is therefore $60 - 6 = 54$ minutes.

3 Problems

1. Julian and Joshua each maintain a constant speed as they run laps around a 400-meter track. In the time it takes Julian to complete two laps, Joshua completes three laps. Julian runs each mile in 12 minutes. How many minutes does it take Joshua to run exactly one mile? (2007 MATHCOUNTS State Countdown)
2. Rico can run 5 miles in the same amount of time that Donna can run 3 miles. Rico runs a rate 4 miles per hour faster than Donna. At that rate, what is the number of miles that Rico runs in 1 hour and 30 minutes? (Alcumus)
3. Cassandra sets her watch to the correct time at noon. At the actual time of 1:00 PM, she notices that her watch reads 12:57 and 36 seconds. Assuming that her watch loses time at a constant rate, what will be the actual time when her watch first reads 10:00 PM? (2003 AMC 12)
4. One complete lap around a particular circular track is 400 meters. Jun and Quan each start running at the starting line and run around the track; Jun runs clockwise at 3 meters per second, and Quan runs counterclockwise at 5 meters per second. When they meet for the sixth time after starting, they stop and both walk back together along the track to the starting line. What is the shortest distance they could walk back on the track together? (2006 MATHCOUNTS State Target)
5. At Pizza Perfect, Ron and Harold make pizza crusts. When they work separately Ron finishes the job of making 100 crusts 1.2 hours before Harold finishes the same job. When they work together they finish making 100 crusts

in 1.8 hours. How many hours, to the nearest tenth of an hour, does it take Ron working alone to make 100 crusts?

6. The students in Mrs. Reed's English class are reading the same 760-page novel. Three friends, Alice, Bob and Chandra, are in the class. Alice reads a page in 20 seconds, Bob reads a page in 45 seconds and Chandra reads a page in 30 seconds.

Before Chandra and Bob start reading, Alice says she would like to team read with them. If they divide the book into three sections so that each reads for the same length of time, how many seconds will each have to read? (2006 AMC 8)

7. Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium? (2007 AMC 10/12)

8. At 9:00 am, an empty water tank begins to be filled with water flowing through a hose at a rate of five gallons per minute. Seven hours later, water flowing through a second hose also starts to fill the tank at a rate of eight gallons per minute. Some time later, the first hose is turned off but the second hose continues to be used to fill the tank. At midnight the 7740-gallon tank is finally full. At what time was the first hose turned off? (MATHCOUNTS)

9. Two trains are approaching one another from opposite directions on parallel tracks. Each train is 150 ft long, but one of the trains is moving at 50 ft/sec, while the other is traveling at only 30 ft/sec. How many seconds elapse from the time the trains first begin to overlap to the time they have completely passed one another?

10. Wilma and Betty ran a 100-meter race at top speed, and Wilma finished when Betty had 10 meters to go. They decided to run again, but this time Wilma gave Betty an advantage. Wilma's starting point was 10 meters behind the original starting point. Given that Wilma and Betty run at the same speeds as the previous race, how many meters will Betty be from the finish line when Wilma crosses it? (MATHCOUNTS)