# Quadratics Handout

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## 1 Introduction

### 1.1 Basics

**Definition 1.1.1.** A quadratic is any expression of the form $ax^2 + bx + c$.

The *roots* of a quadratic are values of $x$ for which the expression equals 0.

**Example 1.1.1.** Find the roots of $x^2 - 3x + 2 = 0$.

*Solution:* This can be done with a simple guess and check. $x = 0$ gives 2, $x = 1$ gives 0, and $x = 2$ gives 0. The two roots are therefore 1 and 2.

There are more sophisticated ways to solve for the roots, particularly *factoring*. The quadratic $x^2 - 3x + 2$ is equivalent to $(x - 1)(x - 2)$, as you can see by expanding this expression. When the quadratic is in this form, it's easy to see how $x = 1$ or $x = 2$ satisfies the equation.

If we have $(x + a)(x + b)$, expanding this out gives $x^2 + (a + b)x + ab$. This lets us factor an equation like $x^2 + sx + t$ by checking through the factors of t and summing them to see if they equal $s$.

**Example 1.1.2.** Solve for the roots of $x^2 + 19x + 60$.

*Solution:* With experience, you can quickly recognize that $19 = 4 + 15$ and that $4 \cdot 15 = 60$. This gives us $x^2 + 19x + 60 = (x + 4)(x + 15)$. The roots are therefore $-4$ and $-15$.

If the constant term is negative, this means that one root is negative.

**Example 1.1.3.** Find the roots of $x^2 + 7x - 60$.

*Solution:* This time, one factor of 60 has to be 7 greater then the other. We know that $12 - 5 = 7$, so we can factor $x^2 + 7x - 60 = (x + 12)(x - 5)$. The roots are therefore -12 and 5.

**Exercise 1.1.1.** Factor $x^2 - 9x + 18$.
**Exercise 1.1.2.** Factor $x^2 + x - 72$.
**Exercise 1.1.3.** Factor $3x^2 - 13x + 4$.
**Exercise 1.1.4.** Factor $\frac{1}{3}x^2 + 5x + \frac{44}{3}$.

## 1.2 Completing the Square

Often, it's impossible to factor certain quadratics. For example, there are no integer roots for $x^2 + 40x + 41$. However, we can still find the precise roots for these equations by **completing the square**. To start, let's examine *binomial squares.*

**Example 1.2.1.** Expand out $(x + 1)^2$ . What about $(x - 1)^2$?

*Solution:* Using the distributive property, we can multiply out $(x+1)(x+1)$ into $x^2+x+x+1 = x^2+2x+1$. We can do much the same thing for $(x - 1)^2$, getting $x^2 - x - x + 1 = x^2 - 2x + 1$.

In general, $(x + a)^2 = x^2 + 2a + a^2$. Let's see what we can do with this.

Let's look at a quadratic like $x^2 + 100x + 2464$. Factoring this would certainly be difficult. However, $(x + 50)^2 = x^2 + 100x + 2500$. We then have $(x + 50)^2 - 36 = x^2 + 100x + 2464$. When trying to solve for the roots, we have $(x + 50)^2 - 36 = 0$. We can rearrange this to $(x + 50)^2 = 36$. By taking the square root of both sides, we have $x + 50 = \pm 6$. It is easy to then find the roots, which are -44 and -56. This is the process of completing the square.

**Exercise 1.2.1.** Solve $x^2 + 22x + 120$ by completing the square.
**Exercise 1.2.2.** Solve $x^2 - 2x - 143$ by completing the square.

When the leading coefficient is not 1, we can still complete the square by dividing.

**Example 1.2.2.** Find the roots of $2x^2 + 17x + 21$.

*Solution:* We can extract a factor of two from $2x^2 + 17x$, turning it into $2\left(x^2 + \frac{17}{2}x\right)$. We can then complete the square, deriving the expression $2\left(x + \frac{17}{4}\right)^2 + 21 - 2 \cdot \left(\frac{17}{4}\right)^2$. To solve for the roots, we have $2\left(x + \frac{17}{4}\right)^2 = \frac{242}{16}$. We divide both sides to get $\left(x + \frac{17}{4}\right)^2 = \frac{121}{16}$, then take the square root to get $x + \frac{17}{4} = \pm\frac{11}{4}$, for the solutions $-\frac{3}{2}$ and 7.

Through completing the square, we can derive a general formula for the roots of a quadratic:

**Definition 1.2.1.** By the *quadratic formula*, a root $r$ of a quadratic $ax^2 + bx + c$ satisfies the following:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Let's go through the derivation.

First, we have

$$ax^2 + bx + c = 0.$$

We divide both sides by $a$ to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Next, we complete the square.

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2},$$

so

$$\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0.$$

Rearranging and giving the right side a common denominator gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

and isolating $x$ gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice the $\sqrt{b^2 - 4ac}$ term in the quadratic formula. We call $b^2 - 4ac$ the *discriminant*. If the discriminant is negative, there are no real solutions to the quadratic.

**Exercise 1.2.3.** What is the square of the difference between the roots of $x^2 - 7x + 10$?

**Exercise 1.2.4.** How many solutions does $5x^2 - 30x + 45$ have?

**Exercise 1.2.5.** Determine if $x^2 + 2x + 7 = 0$ has any solutions.

# 2 Advanced Techniques

## 2.1 Graphing Quadratics

It is possible to graph quadratics on the Cartesian plane, where they form parabolas. The *vertex* of the quadratic is its minimum or maximum point. The points where the parabola intercept the x-axis are the roots of the quadratic. Note that if there are no intersections, the quadratic has no roots.

Finding the vertex of a quadratic is difficult when it is in its $ax^2 + bx + c$ form. However, we can also express quadratics in *vertex form*.

**Definition 2.1.1.** A quadratic $y = a(x - h)^2 + k$ has a vertex at $(h, k)$.

Notice that the $(x - h)^2$ term can never be negative, meaning that its lowest possible value is zero, reached when $x = h$. In addition, when this term is zero, we are left with $y = k$. We already know how to put quadratics into vertex form; it's just completing the square!

**Example 2.1.1.** Find the vertex of $y = x^2 + 6x + 8$.

*Solution:* We complete the square to get $(x+3)^2 - 1 = x^2 + 6x + 8$. The vertex is therefore at $(-3, -1)$.

**Exercise 2.1.1.** Complete the square: $25x^2 + 20x - 10 = 0$.
**Exercise 2.1.2.** For specific positive numbers $m$ and $n$, the quadratics $16x^2 + 36x + 56$ and $(mx + n)^2$ differ only in their constant term. What is $mn$? (Alcumus)
**Exercise 2.1.3.** Find $a$ such that $ax^2 + 12x + 9$ is the square of a binomial.
**Exercise 2.1.4.** When the expression $-5x^2 + 20x - 17$ is written in the form $a(x+d)^2 + e$, where $a$, $d$, and $e$ are constants, then what is the sum $a + d + e$? (Alcumus)

Let's take a look at some more advanced problems.

**Example 2.1.2.** Find $a + b + c$ if the graph of the equation $y = ax^2 + bx + c$ is a parabola with vertex $(5, 3)$, vertical axis of symmetry, and contains the point $(2, 0)$. (Alcumus)

*Solution:* The problem gives us the vertex, so the equation of the parabola looks like $y = a(x-5)^2 + 3$. We know that $(2, 0)$ is a solution, so we plug it in and get $0 = 9a + 3$. It's trivial to find $a = -\frac{1}{3}$. We then expand out $-\frac{1}{3}(x-5)^2 + 3$ to get $-\frac{1}{3}x^2 + \frac{10}{3}x - \frac{16}{3}$. We sum up the coefficients to get $-\frac{7}{3}$.

Notice that if you have a quadratic $ax^2 + bx + c$, if $x = 1$ the equation is equivalent to $a + b + c$. So above, we could have just plugged in $x = 1$ into $-\frac{1}{3}(x-5)^2 + 3$, getting the same answer.

**Example 2.1.3.** Find the sum of the x-coordinates of the points where the parabola $x^2 - 4x - 2$ and the line $y = 3x - 12$ intersect.

*Solution:* We solve for the solutions to $3x - 12 = x^2 - 4x - 2$. This turns it into another quadratic, $0 = x^2 - 7x + 10$. We can use the quadratic formula or factor this into $(x-2)(x-5)$. We sum the roots to get $2+5=\mathbf{7}$.

**Example 2.1.4.** A parabola has its highest point at $(2,1)$ on the Cartesian plane, and passes through the point $(-4,-3)$. What is the distance between the coordinates of its roots?

*Solution:* We know that the vertex is $(2,1)$, so we have $y = a\left(x-2\right)^2 + 1$. Plugging in $(-4,-3)$, we have $-3 = 36a + 1$, so $36a = -4$ and $a$ equals $-\frac{1}{9}$. We now have the quadratic $y = -\frac{1}{9}\left(x-2\right)^2 + 1$, which we can easily solve as $x = 5, -1$. $5 - (-1) = 6$, so we have our answer.

Alternatively, it's possible to find the distance between the roots by using the quadratic formula. One root will be $\frac{-b+\sqrt{b^2-4ac}}{2a}$, while the other will be $\frac{-b-\sqrt{b^2-4ac}}{2a}$. Subtracting the two, we have $\frac{\sqrt{b^2-4ac}}{a}$ as the difference. Once we've arrived at $y = -\frac{1}{9}\left(x-2\right)^2 + 1$, we can derive $a, b$, and $c$. $a$ will be $-\frac{1}{9}$, b will be $\frac{4}{9}$, and $c$ will be $\frac{5}{9}$. This gives us $\frac{\sqrt{\frac{16}{81}+\frac{20}{81}}}{-\frac{1}{9}} = -6$. Therefore, our answer is 6 (since distances must be positive.)

## 2.2 Quadratic Optimization

We can use quadratics to model many situations, and in many cases you will be trying to find the maximum or minimum case. As we covered in the previous section, the maximum or minimum of a quadratic is at its vertex, so many of these problems are simply about deriving and solving an equation.

**Example 2.2.1.** A farmer has 200 feet of fencing, and he is trying to fence a rectangular plot of land. What is the maximum area, and how long should the sides be?

*Solution:* Let's call the length of one side of the fence $x$. Then the perpendicular side must be $100-x$ in order for the perimeter to be 200. So the area of the plot is $x(100-x)$. We expand this out into $-x^2 + 100x$. We can put this into vertex form as $-\left(x-50\right)^2 + 2500$. Therefore, the maximum area is 2500.

Other examples of optimization include finding costs.

**Example 2.2.2.** An airline currently sells 1000 tickets every day for \$50 every day. They know that for every five dollars they raise the price, they will sell 10 fewer tickets. How much money should they charge to make the most money?

*Solution:* Let $x$ be the cost of the ticket. We can express the profit of the airline as $(1000 - 10(x - 50))x$. Expanding this out gives us $1500x - 10x^2$. Once again, we can put this into vertex form as $-10\left(x - 75\right)^2 + 56250$. This shows that the highest profit is made when tickets are 75 dollars.

You might have noticed that for Example 1.4.1, 50 is the average of 0 and 100, and in Example 1.4.2, 75 is the average of the two roots of the quadratic. In fact, the minimum or maximum of a quadratic will always be halfway between the roots. We can see this by looking at the quadratic formula. The average of the two roots $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$ is just $\frac{-b}{2a}$, since the discriminant terms cancel out. As we saw when we converted a quadratic into vertex form, when we plug this in it minimizes the term $\left(x + \frac{b}{2a}\right)^2$ to zero. This lets us solve many optimization problems quite quickly.

**Example 2.2.3.** A farmer is building another pen, and he has 400 ft of fencing. The pen will be rectangular, and he will use a stream as one side of the pen. What is the maximum area he can fence?

*Solution:* Let the length of the side parallel to the stream be $x$. The other two sides will be $200 - \frac{1}{2}x$ ft long, in order for the total amount of fencing to add up to 400. We have the area as $x(200 - \frac{1}{2}x)$, and we can extract $\frac{1}{2}$ to get $\frac{1}{2}x(400 - x)$. The roots are 0 and 400, so $x$ should be $\frac{400}{2}$ for the largest area.

Optimization also comes into play when trying to find distances on a coordinate plane.

**Example 2.2.4.** The smallest distance between the origin and a point on the parabola $y = x^2 - 5$ can be expressed as $\frac{\sqrt{a}}{b}$, where $a$ and $b$ are positive integers, and $a$ is not divisible by the square of any prime. Find $a+b$. (Alcumus)

*Solution:* Recall that the distance between a point $(x, y)$ and the origin is $\sqrt{x^2 + y^2}$. We are supplied the equation $y = x^2 - 5$, so the distance between any point on this quadratic can be expressed as $\sqrt{x^2 + (x^2 - 5)^2}$. We can expand out the terms inside the square root into $x^4 - 9x^2 + 25$. We can make the substitution $a = x^2$ in order to turn this into an quadratic, $a^2 - 9a + 25$. We know that when we take this into vertex form, in the $(a - h)^2$ term, $h$ has to be $\frac{9}{2}$, so we can simply plug in 4.5 into the equation to derive $20.25 - 40.5 + 25 = 4.75$. This is equivalent to $\frac{19}{4}$, so when we take the square root of this, we have $\frac{\sqrt{19}}{2}$ as the distance.