Week 13: Trigonometry

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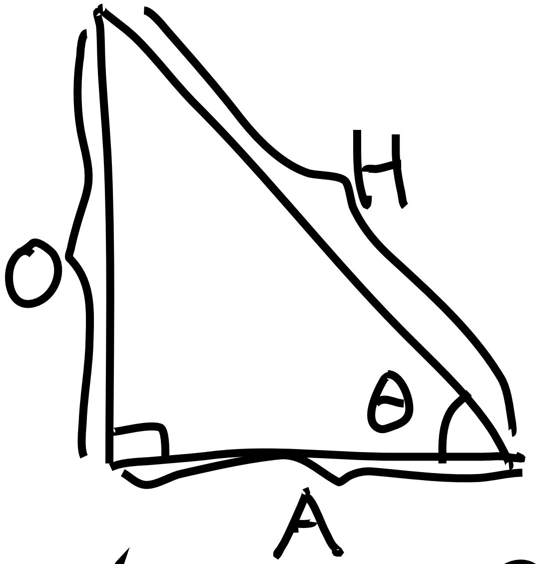
Meeting Plan

1. Trigonometry Presentation
2. Break (5 min.)
3. Trigonometry Kahoot!
4. Exercise Leaderboard Award Ceremony

Definitions



$$\begin{cases} \sin\theta = \dfrac{O}{H} \\ \cos\theta = \dfrac{A}{H} \\ \tan\theta = \dfrac{O}{A} \end{cases}$$

SOHCAHTOA

$\sin x = y$

$x = \sin^{-1} y$

$= \arcsin y$

Cosecant

$\csc\theta = \dfrac{1}{\sin\theta}$

Secant

$\sec\theta = \dfrac{1}{\cos\theta}$

Cotangent

$\cot\theta = \dfrac{1}{\tan\theta}$

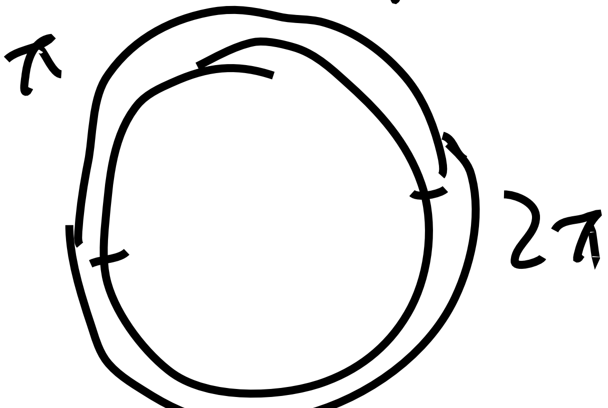
Graphing Trigonometric Functions

In this section, we focus on the graphs of trigonometric functions. We will plot the functions on the standard coordinate plane, except the values on the x-axis represent angles in *radians*. For those who aren't familiar with this, one radian is defined to be equal to $\frac{180}{\pi}$ degrees. This means that π radians is equal to $180°$ and 2π radians is equal to $360°$.

1 rad $= \frac{180}{\pi}°$

π radians $= 180°$

2π rad $= 360°$



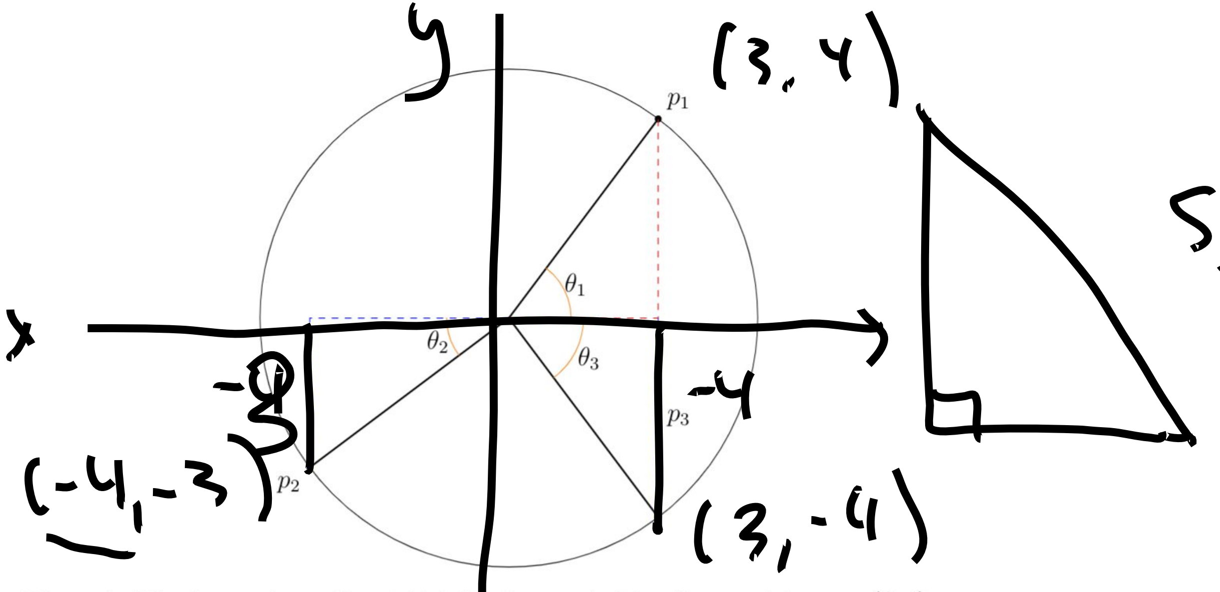


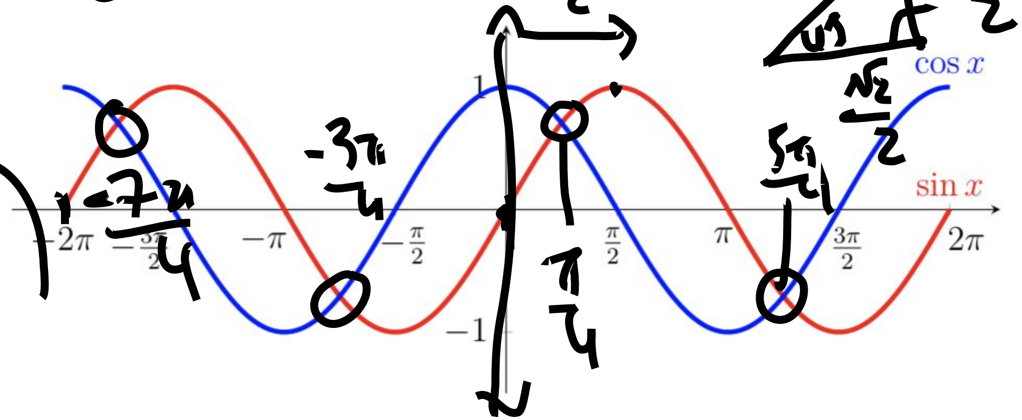
Figure 1: The figure shows the right triangles created by three points $p_1 = (3, 4)$, $p_2 = (-4, -3)$, and $p_3 = (3, -4)$. The legs of these right triangles with non-positive length are colored in blue, and the others are shown in red. We see that $\sin\theta_1 = \sin\theta_2 = \frac{4}{5}$ and $\sin\theta_2 = -\frac{3}{5}$.

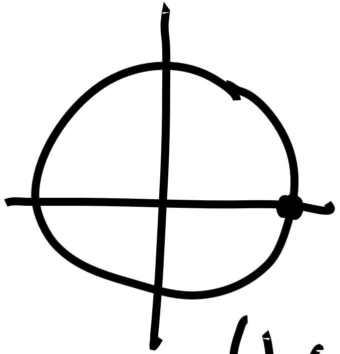
$\sin\theta_1 = \frac{4}{5}$

$\sin\theta_2 = \frac{-3}{5}$

$\sin\theta_3 = \frac{-4}{5}$

period $= 2\pi$





$(1, 0)$

$\left(\frac{\pi}{4} + k\pi, \frac{\sqrt{2}}{2}\right)$

k

Figure 2: The figure shows the plots of $y = \sin x$ (~~in blue~~) and $y = \cos x$ ~~(in red)~~ on the interval $[-2\pi, 2\pi]$.

red

blue

Period π

∞

$-\infty$

$\frac{1}{0}$

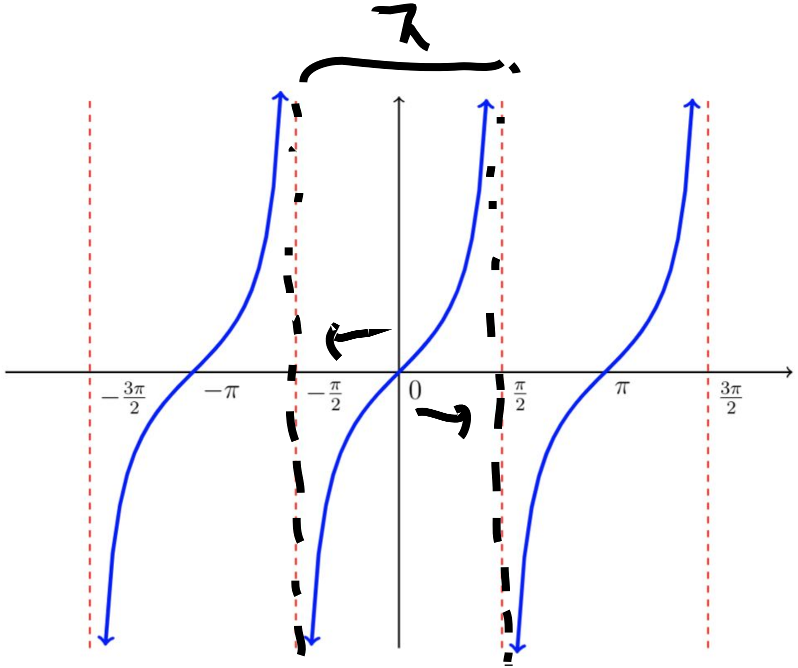
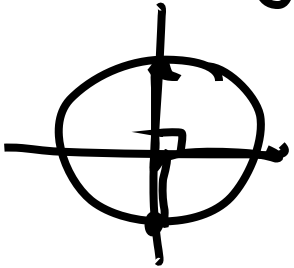


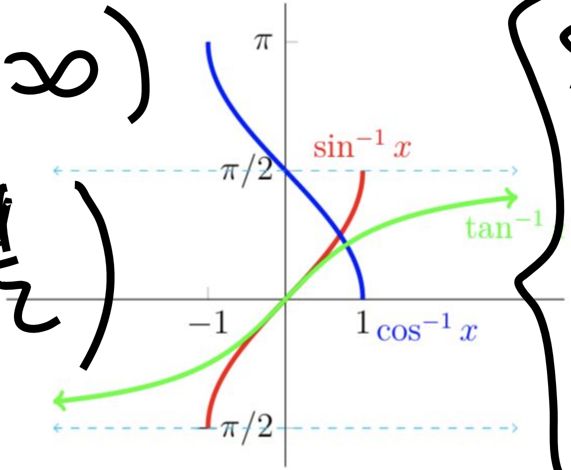
Figure 3: The figure shows the plot of $y = \tan x$ (in blue) on the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.



$$x = \frac{(2k+1)\pi}{2} \qquad \frac{(2k+1)\pi}{2}$$

$\tan^{-1} x : (-\infty, \infty)$

$: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$\begin{cases} \sin^{-1} x & [-1, 1] \\ & \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \cos^{-1} x & [-1, 1] \\ & [0, \pi] \end{cases}$$

Figure 4: The figure shows the plots of $y = \sin^{-1} x$ (in red), $y = \cos^{-1} x$ (in blue), and $y = \tan^{-1} x$ (in green) on the interval $[-1, 1]$.

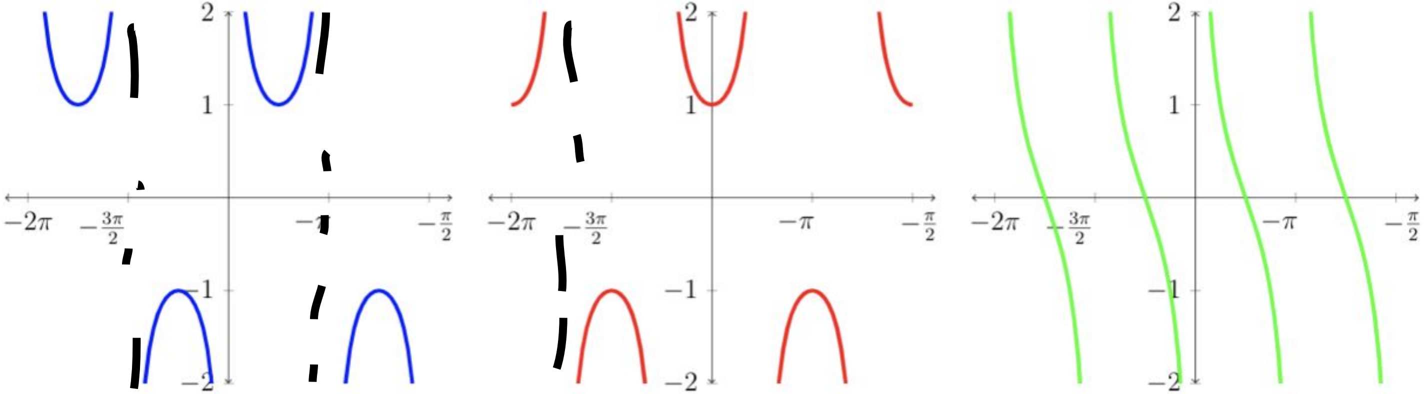
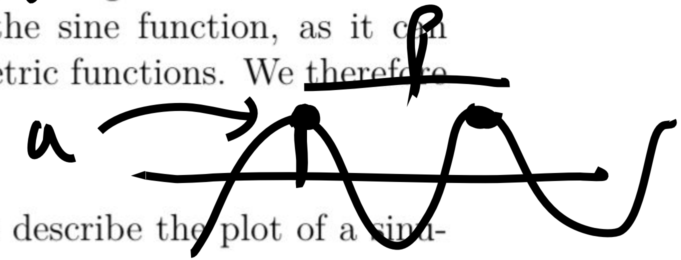


Figure 5: The figure shows the plots of $y = \csc x$ (in blue), $y = \sec x$ (in red), and $y = \cot x$ (in green) on the interval $[-2\pi, 2\pi]$.

Our final topic for this subsection is plotting transformations of trigonometric functions. We will only elaborate on graphing transformations of the sine function, as it can easily be generalized to transformations of other trigonometric functions. We therefore consider the function

$$y = a\sin(bx + c) + d.$$



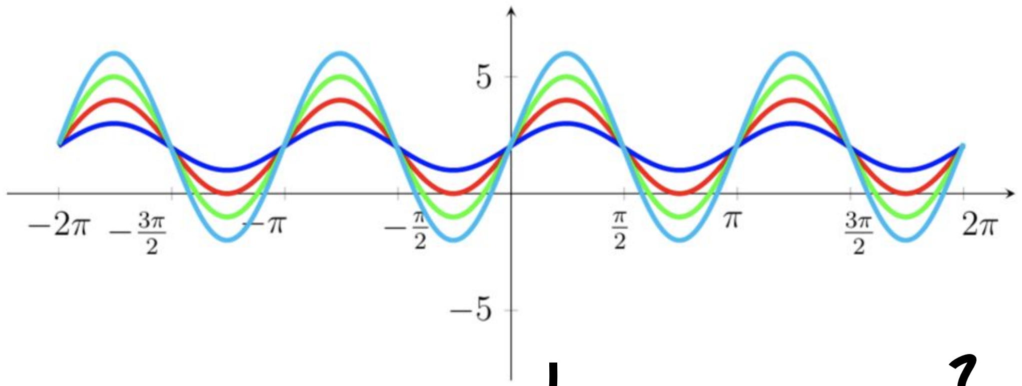
Now, we introduce the following useful terms that help us describe the plot of a sinusoidal function.

- The *amplitude* of a sinusoidal function is the height of each peak from the x-axis.
- The *period* is the distance between peaks, or the length of the repeating portion of the graph.
- The *frequency* of a sinusoidal curve represents how often the function repeats. By this definition, we see that the frequency can be expressed as the reciprocal of the period, and has units of cycles per unit of time (which represents the unit of the x-axis).

A = 440 Hz

440 oscillations /sec

$$y = a\sin(bx + c) + d.$$



1

2

Figure 6: The figure shows the plots of $y = \sin(2x+2)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 3\sin(2x+2)+2$ (in green), and $y = 4\sin(2x+2)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter a. By doing this, we see that increasing a increases the amplitude, which corresponds to a vertical stretch. Likewise, decreasing a decreases the amplitude, vertically shrinking the graph. Thus, we conclude that the a is the parameter representing/determining the amplitude.

$$y = a\sin(bx + c) + d.$$

period: $\dfrac{2\pi}{b}$

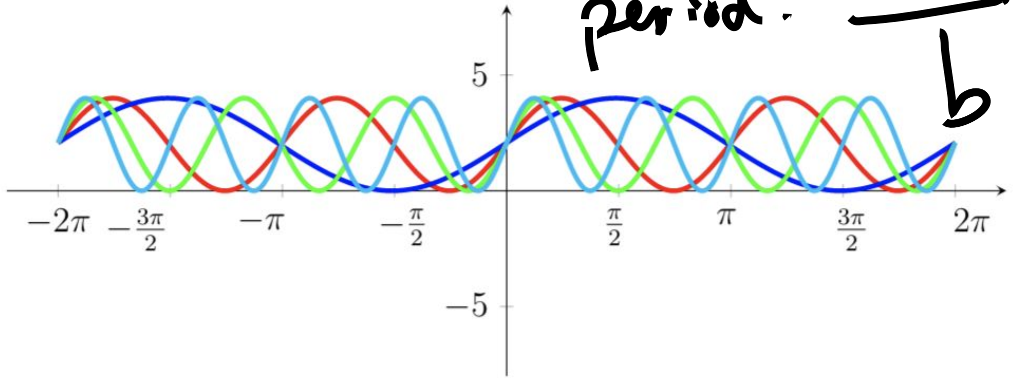
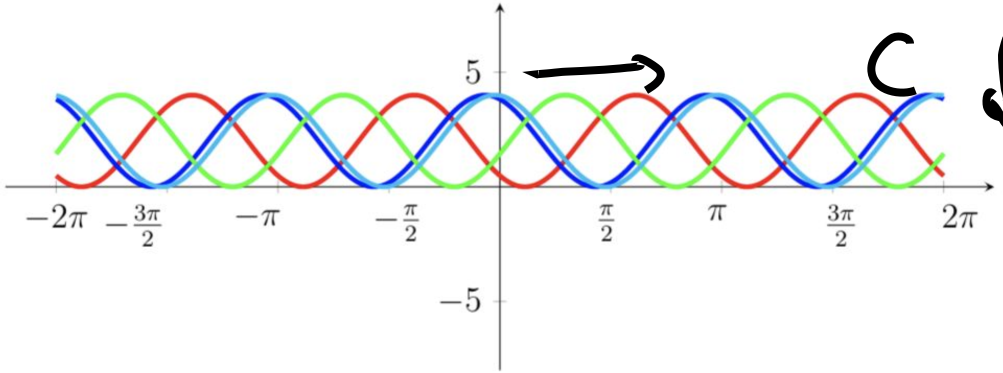


Figure 7: The figure shows the plots of $y = 2\sin(x+2)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(3x+2)+2$ (in green), and $y = 2\sin(4x+2)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter b. By doing this, we see that increasing b decreases the period, which corresponds to a horizontal shrink. Likewise, decreasing b increases the period, horizontally stretching the graph. Thus, we conclude that the b is the parameter representing/determining the period, which is equal to $\frac{2\pi}{b}$ (giving us an inverse relationship between the period and b).

$$y = a\sin(bx + c) + d.$$

c

c



$\frac{c}{a}$ Leftwards

Figure 8: The figure shows the plots of $y = 2\sin(2x+1)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(2x+3)+2$ (in green), and $y = 2\sin(2x+4)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter c. By doing this, we see that increasing c shifts the graph more to the left. Likewise, decreasing c shifts the graph to the right. Notice that these translations leave the period and amplitude unchanged. Thus, we conclude that the c is the parameter representing the phase shift, which is equal to $\frac{c}{a}$ leftwards (as a phase shift of zero means that the function passes through the origin). Note that if $\frac{c}{a}$ is negative, then we have a rightward shift.

$$y = a\sin(bx + c) + d.$$

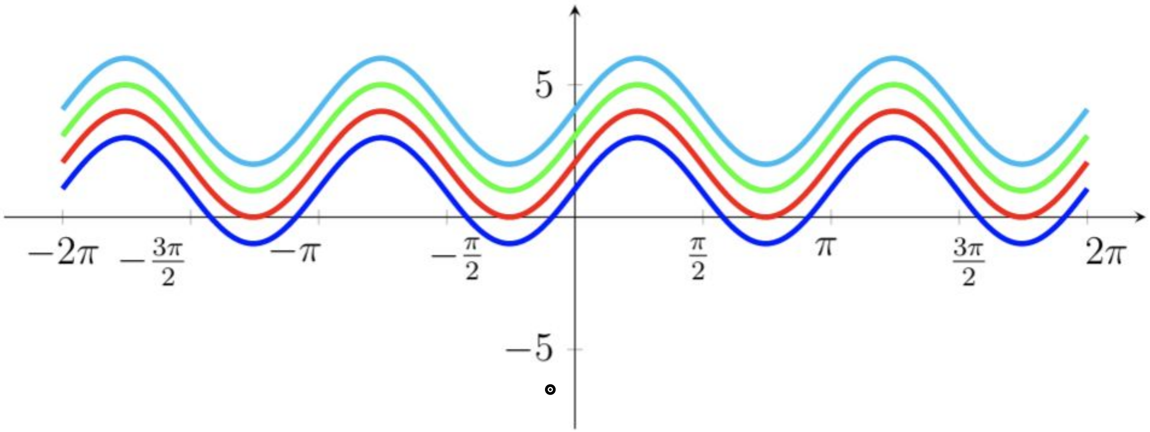
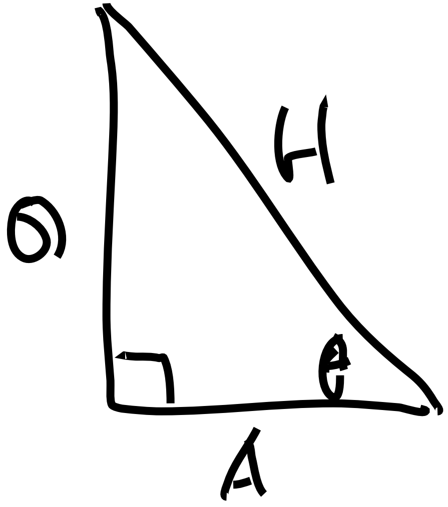


Figure 9: The figure shows the plots of $y = 2\sin(2x+2)+1$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(2x + 2) + 3$ (in green), and $y = 2\sin(2x + 2) + 4$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x + 2) + 2$ and changed the parameter d. By doing this, we see that increasing d shifts the graph upwards. Likewise, decreasing d shifts the graph downwards. Thus, we conclude that the d is the parameter representing the vertical shift.

Simple Trigonometric Identities



$$\sin^2\theta + \cos^2\theta = 1$$

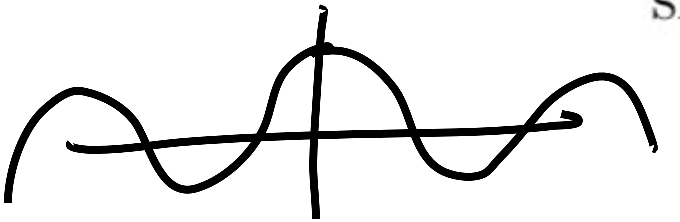
$$O^2 + A^2 = H^2$$

$$\left(\frac{O}{H}\right)^2 + \left(\frac{A}{H}\right)^2 = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

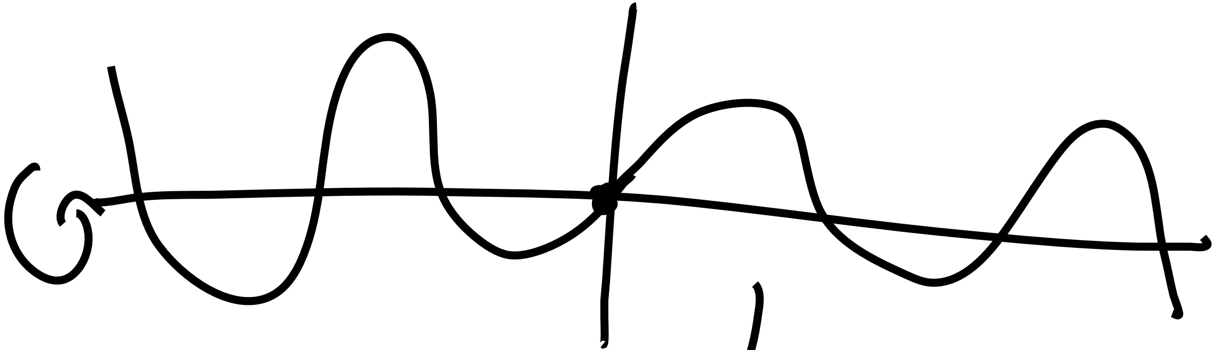
$$1 + \cot^2\theta = \csc^2\theta$$

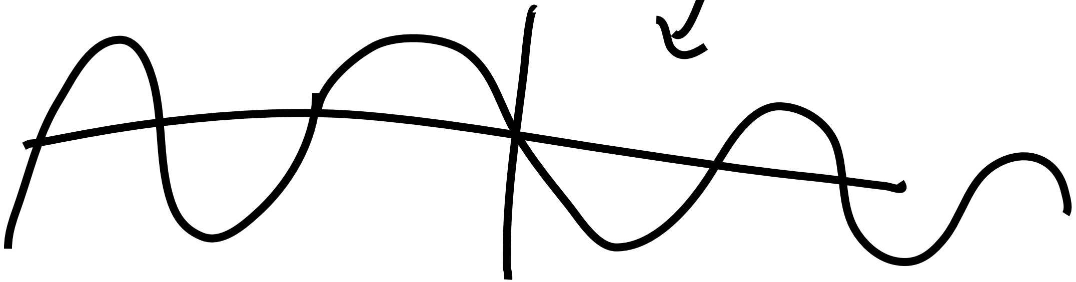
$$\tan^2\theta + 1 = \sec^2\theta$$



$$\sin(-\theta) = -\sin\theta \leftarrow \text{odd}$$

$$\cos(-\theta) = \cos\theta \leftarrow \text{even}$$



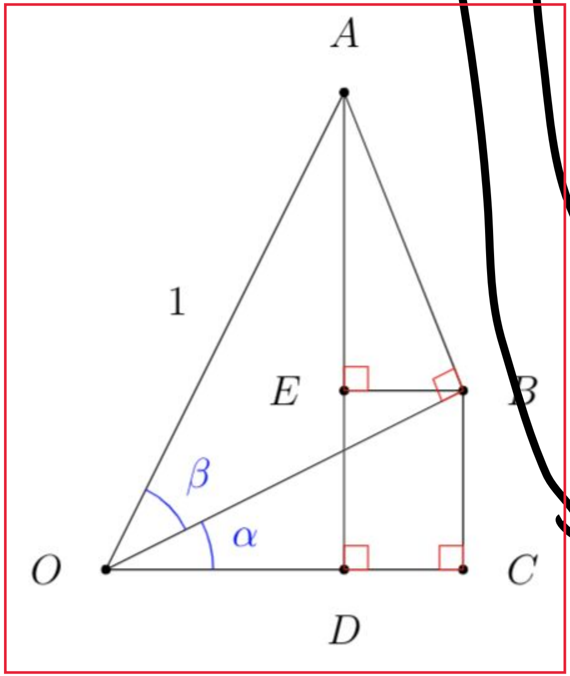


$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

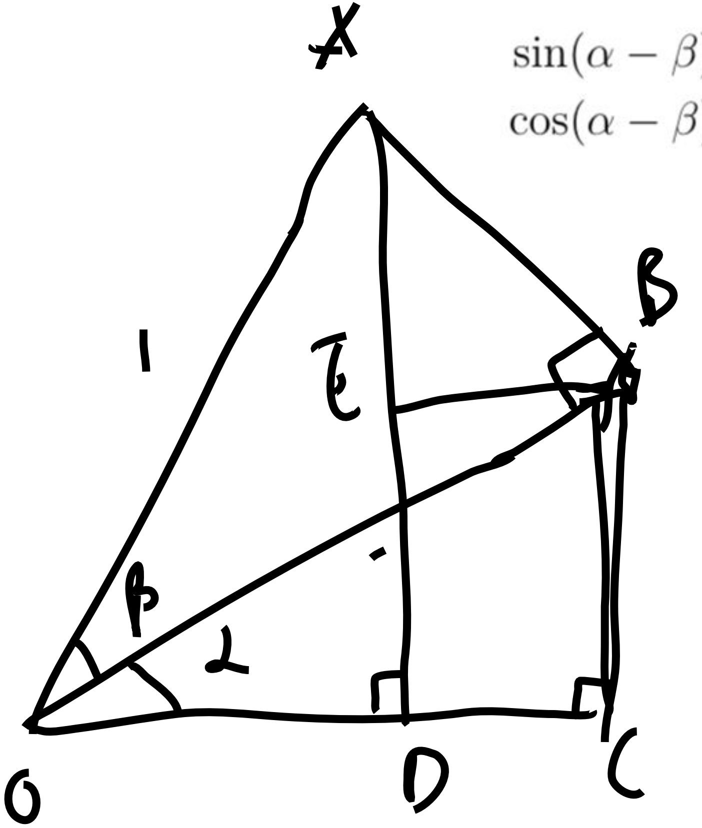
$$e^{\theta i} = \operatorname{cis}\theta$$

$$e^{(\alpha+\beta)i} = \underline{\cos(\alpha+\beta)} + i\,\underline{\sin(\alpha+\beta)}$$

$$e^{\alpha i}\cdot e^{\beta i} = (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta)$$



$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$$



$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = DE + AE$$

$$= \frac{DE}{OB}\cdot OB + \frac{AE}{AB}\cdot AB$$

$$= \sin\alpha\cos\beta$$

$$+ \sin\beta\cos\alpha$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$\sin(\theta \pm \frac{\pi}{2}) = \pm \cos\theta, \quad \sin(\theta \pm \pi) = -\sin\theta, \quad \sin(\theta \pm 2\pi) = \sin\theta$$
$$\cos(\theta \pm \frac{\pi}{2}) = \mp \sin\theta, \quad \cos(\theta \pm \pi) = -\cos\theta, \quad \sin(\cos \pm 2\pi) = \cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin(\theta+\theta)$$

$$\cos(\theta+\theta)$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\frac{\sin\theta}{1+\cos\theta}$$

$$\sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \ldots$$

$$= \sin\theta$$

More Trigonometric Identities

Sum to product identities

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\sin\beta\cos\alpha = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

Checkpoint 4.1. Show that

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

$$\cancel{\cos\alpha\cos\beta} + \sin\alpha\sin\beta$$

$$-\left(\cancel{\cos\alpha\cos\beta} - \sin\alpha\sin\beta\right)$$

$$2\sin\alpha\sin\beta$$

sum to product

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha + \sin\beta = \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)$$

Checkpoint 4.2. Show that

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

$$\cos\alpha + \cos\beta = \cos\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)$$

$$+\cos\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)$$

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

Checkpoint 4.3. Show that

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

$3\theta = 2\theta + \theta$

$\sin 2\theta \cos\theta + \sin\theta \cos 2\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta(1 - 2\sin^2\theta)$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1 - 2\sin^2\theta)$

$=$

$=$

$\cos^2\theta + \sin^2\theta = 1$

$\sin(4\theta)$

$2\sin 2\theta\cos 2\theta$

$2\cdot 2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

Given $\underline{\alpha+\beta+\gamma=\pi}$, $\underbrace{\tan\alpha+\tan\beta+\tan\gamma}=\underbrace{\tan\alpha\tan\beta\tan\gamma}$

180°

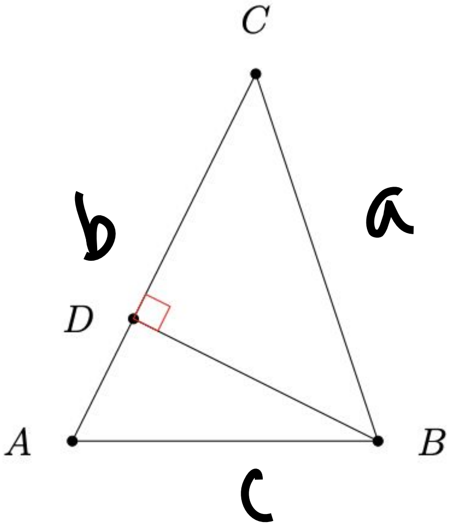
Trigonometric Functions in Geometry

Theorem 5.1. *(Law of Cosines) Let ΔABC be a triangle with side lengths a, b, and c, and let the measure of the angle opposite the side with length c be denoted by C. Then, we have*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

LoC

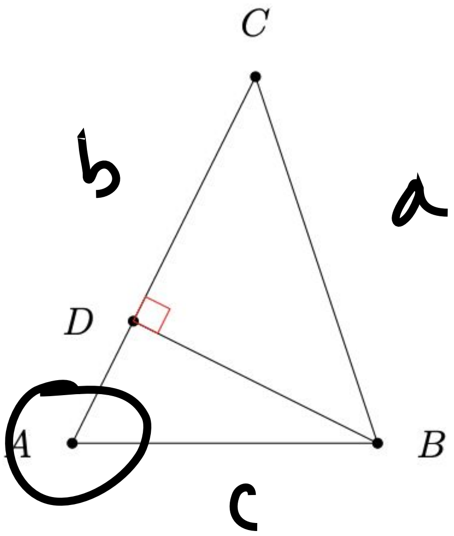
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Theorem 5.2. *Let ΔABC be a triangle with side lengths $BC = a$, $AC = b$, $AB = c$, and circumradius R. Then*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

LoS

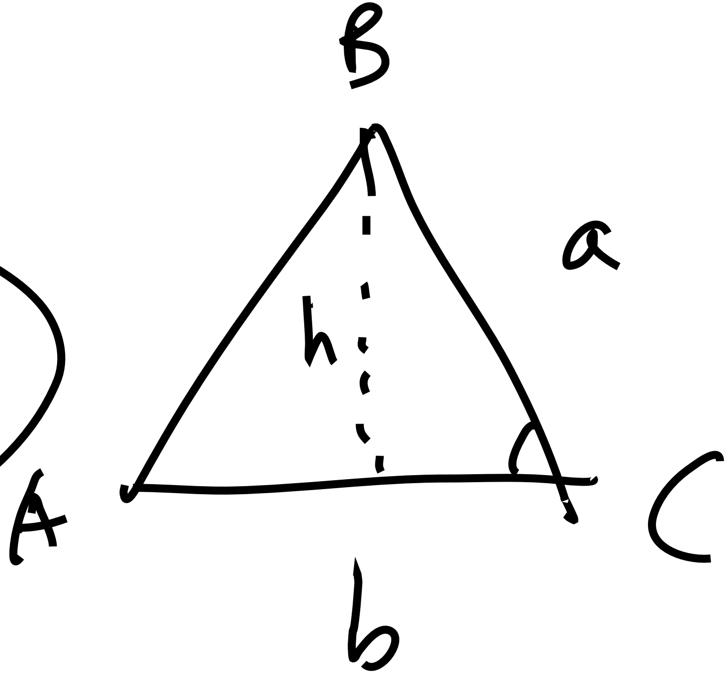


$$\frac{a}{\sin A} =$$

Theorem 5.3. *Let ΔABC be a triangle with legs with length a, b, and c, and the angle in between them measuring C. Then the area of ΔABC is $\frac{1}{2}ab\sin C$.*

$$\frac{1}{2}bh$$

$$\frac{1}{2}ba\sin C$$



$$\sin C = \frac{h}{a}$$

θ	$\sin\theta$	$\cos\theta$
$0°$	0	1
$15°$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
$18°$	$\frac{\sqrt{5}-1}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$36°$	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{\sqrt{5}+1}{4}$
$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$72°$	$\sqrt{\frac{5+\sqrt{5}}{8}}$	$\frac{\sqrt{5}-1}{4}$
$75°$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
$90°$	1	0

Problem Solving in Trigonometry

Example 6.1. At how many points do the graphs $y = \sin x$ and $y = \cos x$ intersect in the interval $[-4\pi, 4\pi]$?

Example 6.2. Show that

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}.$$

Checkpoint 6.1. Show that

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}.$$

Example 6.3. In triangle ABC, $AB = 4$, $BC = 6$, and $AC = 8$. Squares $ABQR$ and $BCST$ are drawn external to and lie in the same plane as ΔABC. Compute QT.
Source: ARML