Iowa City Math Circle Handouts
Trigonometry

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1 Definitions

In this section, we will define the six fundamental trigonometric functions along with their inverses. The reader is likely to be familiar with these definitions, in which case they can skip them, but we will present them for the sake of clarity and completeness.

For any right triangle, let $x \neq 90°$ be an angle of the triangle. Furthermore, let a be the length of the leg opposite x, b be the length of the leg adjacent to x, and c be the length of the hypotenuse. Then we define $\sin x = \frac{a}{c}$, $\cos x = \frac{b}{c}$, and $\tan x = \frac{a}{b}$. In other words, the sine of an angle is the ratio of the length of the opposite leg to the length of the hypotenuse, the cosine of an angle is the ratio of the length of the adjacent leg to the hypotenuse, and the tangent of an angle is the ratio of the length of the opposite leg to the length of the adjacent leg. This is commonly remembered by the acronym "SOHCATOA".

Next, we define $\csc x = \frac{1}{\sin x}$, $\sec x = \frac{1}{\cos x}$, and $\cot x = \frac{1}{\tan x}$. These definitions are useful for algebra involving the standard trigonometric functions. Finally, we define the inverse trigonometric functions. For example, if $\sin x = y$, then we say $x = \sin^{-1} y = \arcsin y$. This extends to the other five basic trigonometric functions as well.

Notice that in a degenerate right triangle, the hypotenuse is equal in length to one leg, so both the sine and cosine functions have a maximum value of 1. This will help us determine the domains of trigonometric functions.

In the next section, we'll see that trigonometric functions can actually also take on negative values, as the sides of a triangle can have negative length based on their orientation with respecting to the coordinate axes.

2 Graphing Trigonometric Functions

In this section, we focus on the graphs of trigonometric functions. We will plot the functions on the standard coordinate plane, except the values on the x-axis represent angles in *radians*. For those who aren't familiar with this, one radian is defined to be equal to $\frac{180}{\pi}$ degrees. This means that π radians is equal to 180° and 2π radians is equal to 360°.

First, we consider the graph of the function $y = \sin x$. Its domain is all real numbers and its range is -1 to 1. From the definitions in the previous section, we can't see how the function can take on negative values. However, let's see how this makes sense by the following "picture". Suppose we choose a circle centered at the origin and any point on that circle, also drawing the radius to that point. Then we form a right triangle with the radius in such a way that one leg lies on the x-axis and the other is parallel to the y-axis. Some examples are shown in Figure 13.1.

Now, if the leg on the x-axis lies on the negative portion of the axis, its given a negative length; otherwise its length is positive. Likewise, if the leg parallel to the y-axis contains points with non-positive y coordinates, the leg is associated with a negative length. Using this notion of edge length, we see how the trigonometric functions, including sine, can produce negative values. It's easy to see that the maximum absolute value of $\sin x$ is 1, because a leg of a right triangle is always shorter than the hypotenuse.

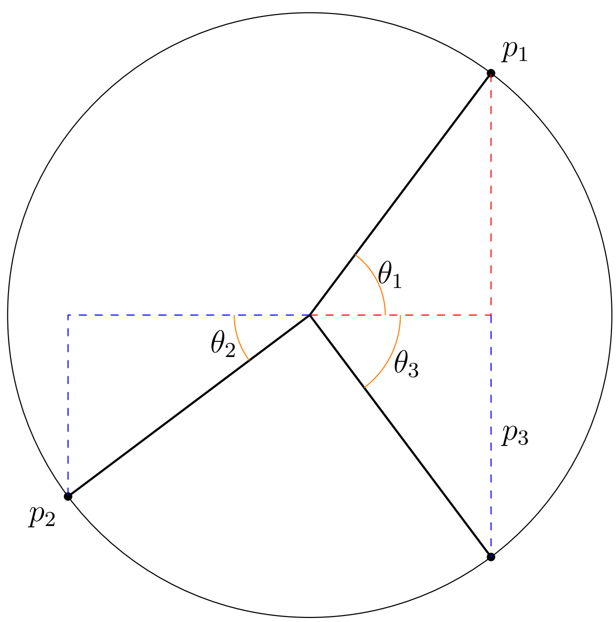


Figure 1: The figure shows the right triangles created by three points $p_1 = (3, 4)$, $p_2 = (-4, -3)$, and $p_3 = (3, -4)$. The legs of these right triangles with non-positive length are colored in blue, and the others are shown in red. We see that $\sin\theta_1 = \sin\theta_2 = \frac{4}{5}$ and $\sin\theta_2 = -\frac{3}{5}$.

Its also easy to see that the domain and range of $y = \cos x$ are the same as $y = \sin x$.

We plot them on the same axis to show their similarities.

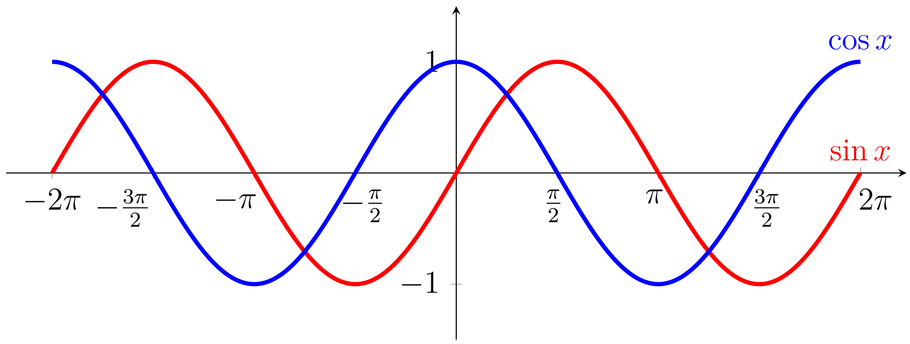


Figure 2: The figure shows the plots of $y = \sin x$ (in blue) and $y = \cos x$ (in red) on the interval $[-2\pi, 2\pi]$.

From these graphs, we make several other observations.

- Both $\sin x$ and $\cos x$ are periodic, and have a *period* (the length of the smallest interval that the function repeats) of 2π units.
- The graph of $y = \sin x$ is essentially a shift of $y = \cos x$ to the right by $\frac{\pi}{2}$ units. This observation comes in handy because it provides us with a useful trigonometric identity, as discussed in the following section/
- The two curves only intersect at the points of the form $\left(\frac{\pi}{4} + k\pi, \frac{\sqrt{2}}{2}\right)$ for all integers k. This occurs precisely when we have a 45-45-90 isosceles right triangle.
- The graph of $y = \cos x$ is symmetric across the y-axis, giving us an important identity we will use in the next section.

Next, we discuss the plot of $\tan x$. As we will see shortly, it has some commonalities with the graphs of the sine and cosine functions in terms of its periodic nature, but also is quite different because of its asymptotic properties.

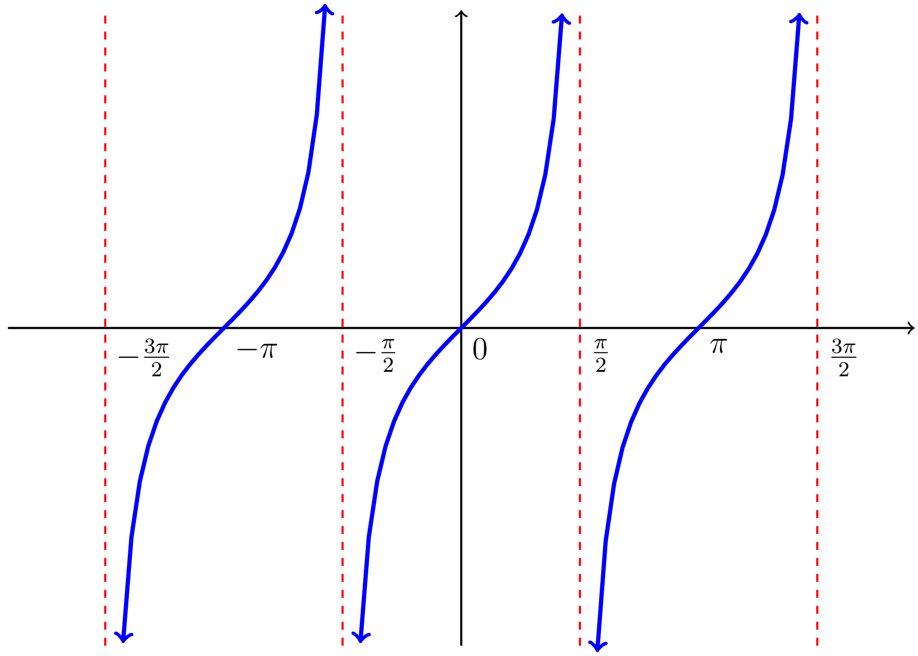


Figure 3: The figure shows the plot of $y = \tan x$ (in blue) on the interval $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

Now, we will complete our analysis of the tangent function by analyzing some key features of its graph.

- It's periodic with a period of π. The cycles of the graph are centered on $k\pi$, for all integers k.
- The graph contains the asymptotes $x = \frac{(2k+1)\pi}{2}$ for all integers k. Moreover, $\tan x$ approaches ∞ as x approaches the asymptote from the left, and $\tan x$ approaches $-\infty$ as x approaches the asymptote from the right. From this, we can also conclude that when $x = \frac{(2k+1)\pi}{2}$, the tangent function is undefined. This is expected since

$$\tan\left(\frac{(2k+1)\pi}{2}\right) = \frac{\sin\left(\frac{(2k+1)\pi}{2}\right)}{\cos\left(\frac{(2k+1)\pi}{2}\right)} = \frac{\sin\left(\frac{(2k+1)\pi}{2}\right)}{0} = \text{undefined}.$$

From Figure 13.1 and the graphs of the singe, cosine, and tangent functions, we see that

- In Quadrant I, all of the three functions are non-negative.
- In Quadrant II, only the sine function is non-negative.
- In Quadrant III, only the tangent function is non-negative.
- In Quadrant IV, only the cosine function is non-negative.

These are very useful in not only determining the sign of certain trigonometric expressions but also to help us derive identities in the following section. One may find the pneumonic "**A**ll **S**ilver **T**ea **C**ups" useful to remember in which quadrants these three functions are non-negative.

Next, we briefly discuss the graphs of the inverse functions of sine, cosine, and tangent. The most important thing to grab from these graphs are their domains and ranges.

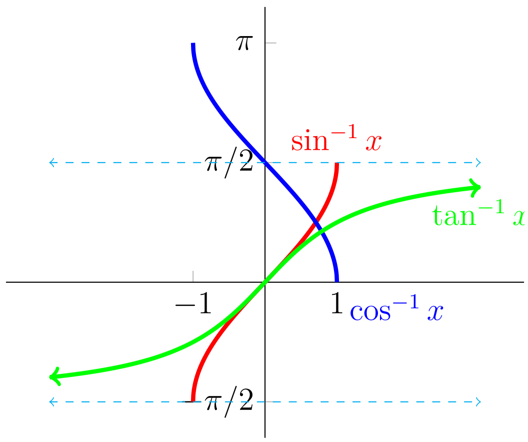


Figure 4: The figure shows the plots of $y = \sin^{-1} x$ (in red), $y = \cos^{-1} x$ (in blue), and $y = \tan^{-1} x$ (in green) on the interval $[-1, 1]$.

From the above graph, we said the domains of $\sin^{-1} x$ and $\cos^{-1} x$ are $[-1, 1]$, and the domain of $\tan^{-1} x$ is $[-\infty, \infty]$. Furthermore, the range of $\sin^{-1} x$ and $\tan^{-1} x$ are $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the range of $\cos^{-1} x$ is $[0, \pi]$. Now, one may question how we obtained these curves from the parent trigonometric functions. They can be obtained by reflecting a half-period across the line $y = x$ (which provides us with the inverse).

Finally, for completeness, we include the graphs of $\csc x$, $\sec x$, and $\cot x$. As expected, they are periodic and have asymptotic behavior.

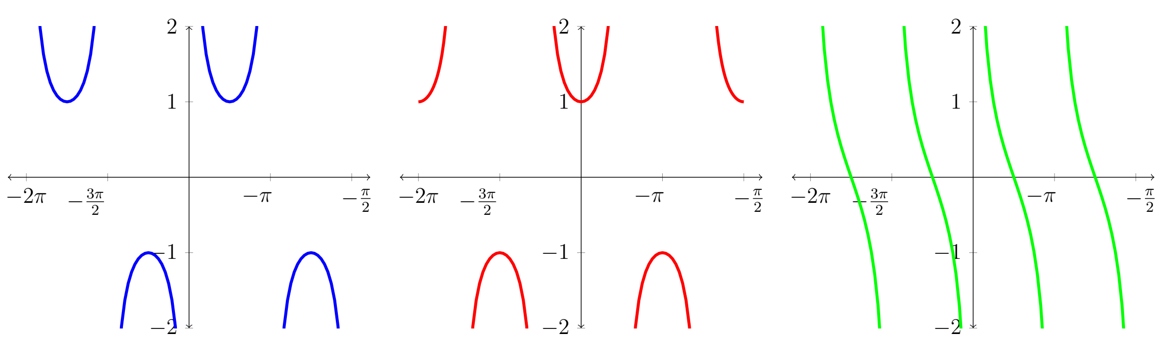


Figure 5: The figure shows the plots of $y = \csc x$ (in blue), $y = \sec x$ (in red), and $y = \cot x$ (in green) on the interval $[-2\pi, 2\pi]$.

Our final topic for this subsection is plotting transformations if trigonometric functions. We will only elaborate on graphing transformations of the sine function, as it can easily be generalized to transformations of other trigonometric functions. We therefore consider the function

$$y = a\sin(bx + c) + d.$$

Now, we introduce the following useful terms that help us describe the plot of a sinusoidal function.

- The *amplitude* of a sinusoidal function is the height of each peak from the x-axis.
- The *period* is the distance between peaks, or the length of the repeating portion of the graph.
- The *frequency* of a sinusoidal curve represents how often the function repeats. By this definition, we see that the frequency can be expressed as the reciprocal of the period, and has units of cycles per unit of time (which represents the unit of the x-axis).

We plot several functions of this form to give you an idea of how a, b, c, and d impact the graph.

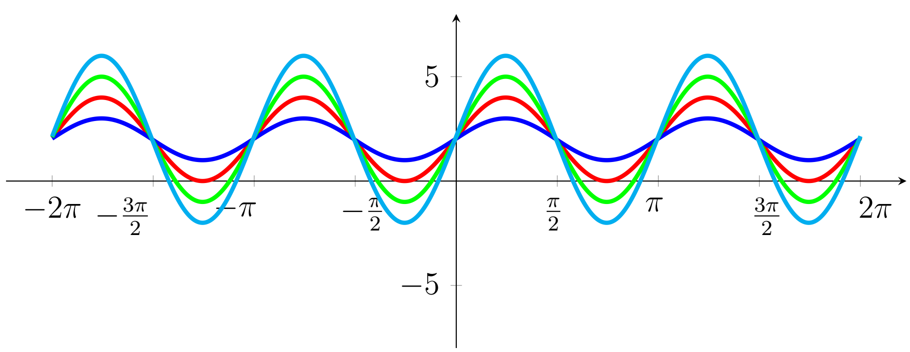


Figure 6: The figure shows the plots of $y = \sin(2x+2)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 3\sin(2x+2)+2$ (in green), and $y = 4\sin(2x+2)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter a. By doing this, we see that increasing a increases the amplitude, which corresponds to a vertical stretch. Likewise, decreasing a decreases the amplitude, vertically shrinking the graph. Thus, we conclude that the a is the parameter representing/determining the amplitude.

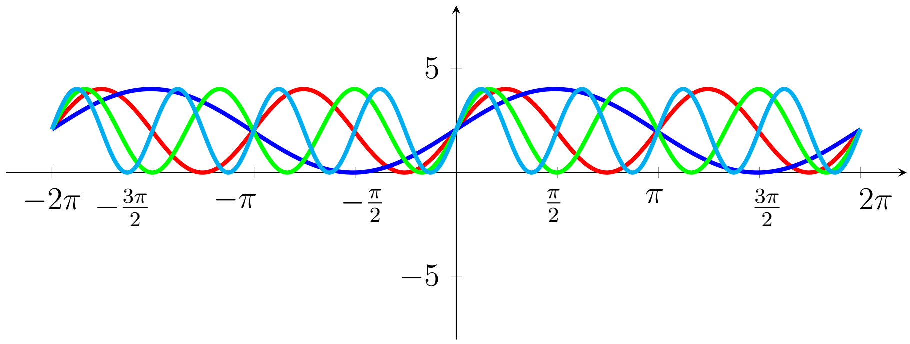


Figure 7: The figure shows the plots of $y = 2\sin(x+2)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(3x+2)+2$ (in green), and $y = 2\sin(4x+2)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter b. By doing this, we see that increasing b decreases the period, which corresponds to a horizontal shrink. Likewise, decreasing b increases the period, horizontally stretching the graph. Thus, we conclude that the b is the parameter representing/determining the period, which is equal to $\frac{2\pi}{b}$ (giving us an inverse relationship between the period and b).

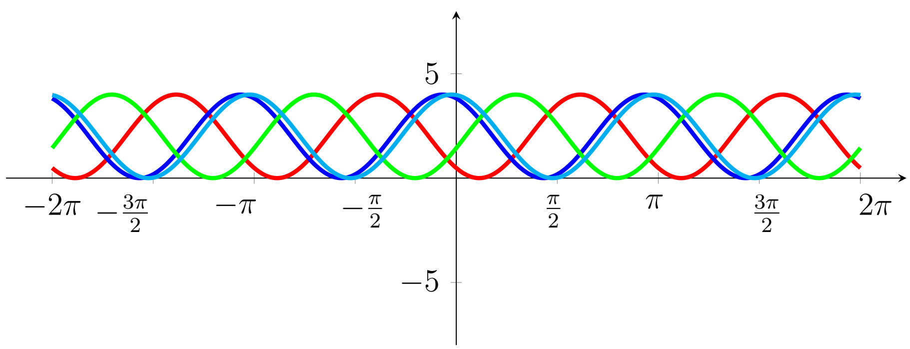


Figure 8: The figure shows the plots of $y = 2\sin(2x+1)+2$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(2x+3)+2$ (in green), and $y = 2\sin(2x+4)+2$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter c. By doing this, we see that increasing c shifts the graph more to the left. Likewise, decreasing c shifts the graph to the right. Notice that these translations leave the period and amplitude unchanged. Thus, we conclude that the c is the parameter representing the phase shift, which is equal to $\frac{c}{a}$ leftwards (as a phase shift of zero means that the function passes through the origin). Note that if $\frac{c}{a}$ is negative, then we have a rightward shift.

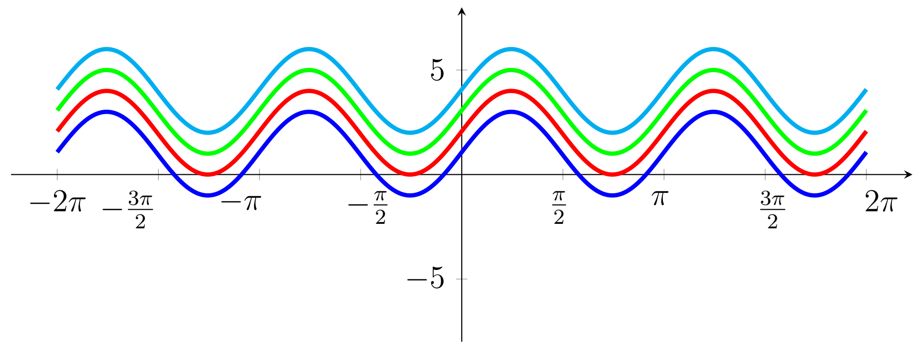


Figure 9: The figure shows the plots of $y = 2\sin(2x+2)+1$ (in blue), $y = 2\sin(2x+2)+2$ (in red), $y = 2\sin(2x+2)+3$ (in green), and $y = 2\sin(2x+2)+4$ (in light blue). Moreover, we started with the base function $y = 2\sin(2x+2)+2$ and changed the parameter d. By doing this, we see that increasing d shifts the graph upwards. Likewise, decreasing d shifts the graph downwards. Thus, we conclude that the d is the parameter representing the vertical shift.

3 Simple Trigonometric Identities

We start with the following classic identity, which serves as the foundation for many other results. In this section, we will frequently reference identities by their symbolic name (e.g. (P)). For completeness, we will give a proof for all the identities. You should know these proofs, so you can derive it during an exam if you forget the identity. However, it's likely you won't have time to do this on real exams so you should try to memorize these identities.

$$\sin^2\theta + \cos^2\theta = 1 \tag{P}$$

Proof. Let h be the hypotenuse of a right triangle with legs of length a and o, and angle opposite the leg with length o to be θ. Then, it suffices to show

$$\left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 = 1 \iff o^2 + a^2 = h^2.$$

This is simply the Pythagorean Theorem, so we finish the proof by simply traversing these steps backwards. □

Now, we can get the following identities by dividing (P) by $\sin^2\theta$ and $\cos^2\theta$, respectively.

$$1 + \cot^2\theta = \csc^2\theta \tag{$P1$}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{$P2$}$$

Some more simple but useful results sum up the odd and evenness of the sine and cosine functions, respectively, as follows.

$$\sin(-\theta) = -\sin\theta \tag{O}$$

$$\cos(-\theta) = \cos\theta \tag{E}$$

Proof. (E) can easily be seen by noticing that $y = \cos x$ is symmetric across the y-axis, and (O) can be seen by noting that the graph of $y = \sin x$ can be rotated 180° to produce the same curve. Both of these can also be observed from Figure 13.1, where the x-component represents $\cos\theta$ and the y-component represents $\sin\theta$. □

Next, we state and prove the sine and cosine addition formulas in multiple ways. These identities are really useful and help us derive other more, complicated identities.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{AS}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{AC}$$

Proof. We first prove this result using by complex numbers and Euler's theorem. From this, we have

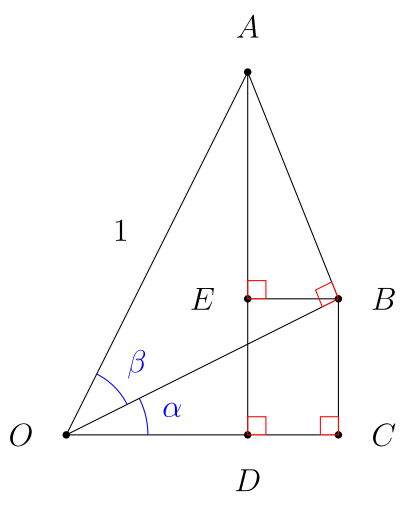
$$e^{(\alpha+\beta)i} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

and

$$\begin{aligned} e^{(\alpha+\beta)i} &= e^{\alpha i}\cdot e^{\beta i} \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha). \end{aligned}$$

Thus, we can equate the imaginary and real parts of $e^{(\alpha+\beta)i}$ in our first and second equations to get the desired results.

Next, we present a geometric proof. The diagram that accompanies the proof is shown below.



Note that we can create the above diagram for any angles $\alpha < \beta$, regardless of whether they are acute or obtuse. From angle chasing using the right angles from our constructions, we have that $\angle BAE = \alpha$. Using this, we see that

$$\sin\alpha = \frac{BC}{OB} = \frac{DE}{OB} = \frac{EB}{AB}, \quad \sin\beta = AB, \quad \cos\alpha = \frac{AE}{AB} = \frac{OC}{OB}, \quad \text{and} \quad \cos\beta = OB.$$

Hence,

$$\begin{aligned}
\sin(\alpha+\beta) &= AD \\
&= DE + EA \\
&= \frac{DE}{OB} \cdot OB + \frac{AE}{AB} \cdot AB \\
&= \sin\alpha\cos\beta + \sin\beta\cos\alpha
\end{aligned}$$

and

$$\begin{aligned}
\cos(\alpha+\beta) &= OD \\
&= OC - DC \\
&= \frac{OC}{OB} \cdot OB - \frac{EB}{AB} \cdot AB \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta,
\end{aligned}$$

as desired. □

From these addition identities, we can easily get the following subtraction identities by substituting $-\beta$ in for β and using the odd-even properties of sine and cosine (namely (O) and (E)).

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha \tag{SS}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{SC}$$

Now, using the sine/cosine-addition/subtraction identities, we can get and prove the corresponding identities for the tangent function.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{AT}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{ST}$$

Proof. Using the addition identities for sine and cosine, we have

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\
&= \frac{\left(\frac{\sin\alpha\cos\beta+\sin\beta\cos\alpha}{\cos\alpha\cos\beta}\right)}{\left(\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)} \\
&= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},
\end{aligned}$$

proving (AT). To prove (ST), we can substitute $-\beta$ in for β and use the odd-even identities. □

Armed with the trigonometric addition identities, we can prove the following identities that establish the cyclic nature of trigonometric functions.

$$\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta, \quad \sin(\theta \pm \pi) = -\sin\theta, \quad \sin(\theta \pm 2\pi) = \sin\theta \tag{CS}$$

$$\cos(\theta \pm \frac{\pi}{2}) = \mp\sin\theta, \quad \cos(\theta \pm \pi) = -\cos\theta, \quad \cos(\theta \pm 2\pi) = \cos\theta \tag{CC}$$

Proof. Apply the trigonometric addition/subtraction formulas. □

Next, we discuss the double-angle identities.

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{DS}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{DC}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{DT}$$

(1)

Proof. (DS) and (DT) are trivial cases of the trigonometric addition formulas with $\alpha = \beta = \theta$. To prove (DC), we again use the cosine addition formula to get $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, but then we apply the Pythagorean trigonometric identity (P) to get the expression in terms of either all sines or all cosines. □

Another set of identities that relate to the double-angle identities are the half-angle identities.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \tag{HS}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \tag{HC}$$

$$\tan\frac{\theta}{2} = \pm\frac{\sin\theta}{1+\cos\theta} \tag{HT}$$

Proof. Substituting $\frac{\theta}{2}$ for θ in the double-angle identity for cosine, we get

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1.$$

This can be easily manipulated to get (HC). We can also substitute in

$$\sin^2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2},$$

a reformulation of the Pythagorean identity, to the equation to get

$$\cos\theta = 2 - 2\sin^2\frac{\theta}{2} - 1.$$

Again, this quadratic can also be manipulated to get (HS). Now, using (HS) and (HC), we have

$$\begin{aligned}
\tan\frac{\theta}{2} &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\
&= \pm\frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} \\
&= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \pm\sqrt{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right)} \\
&= \pm\sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}} \\
&= \pm\frac{\sin\theta}{1+\cos\theta},
\end{aligned}$$

as desired. Note that the plus-or-minus symbol in front of the expressions indicates that we need to choose the angle/roots of the quadratic properly. □

4 More Trigonometric Identities

The next set of trigonometric identities we discuss are called the product-to-sum identities, because they give us a simple way to transform a product of trigonometric functions into a sum.

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta) \tag{$PS1$}$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta) \tag{$PS2$}$$
$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta) \tag{$PS3$}$$
$$2\sin\beta\cos\alpha = \sin(\alpha+\beta) - \sin(\alpha-\beta) \tag{$PS4$}$$

Proof. By applying the trigonometric addition/subtraction identities to each of the terms of the right hand sides of the identities, we can obtain the left-hand sides with a little bit of manipulation. We leave the rest of the smaller details for the reader to work through and as a checkpoint. □

Checkpoint 4.1. Show that

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

Next, we introduce the sum-to-product results, which serve a similar purpose as the previous identities.

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right) \tag{$SP1$}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{$SP2$}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{$SP3$}$$

Proof. We only prove $(SP1)$, as the proofs of the other sum-to-product identities are quite similar. From the sine addition and subtraction identities, we see that

$$\begin{aligned}\sin\alpha + \sin\beta &= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)\\ &= 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right),\end{aligned}$$

as desired. □

Checkpoint 4.2. Show that

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Our next two identities are called the triple-angle identities, as they are just a simple extension of the double-angle formulas.

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta \tag{TS}$$

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta \tag{TC}$$

Proof. To prove these identities, first apply the sin/cosine addition formulas using $3\theta = 2\theta + \theta$, and then use the double angles formulas to simplify the terms that include $\sin(2\theta)$ or $\cos(2\theta)$. Further simplification gives us the desired results. We'll leave the details to the reader and the following checkpoint. □

Checkpoint 4.3. Show that

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta.$$

Our final identity is called the triple-tangent identity. It gives us a way to simplify the product of the tangent of three angles, given that these angles are part of the same triangle. Therefore, it is be quite useful when we are applying trigonometric techniques in geometry.

$$\text{Given} \quad \alpha+\beta+\gamma=\pi, \quad \tan\alpha+\tan\beta+\tan\gamma = \tan\alpha\tan\beta\tan\gamma \tag{TT}$$

Proof. Using the identity $\tan(\pi - \theta) = -\tan\theta$ (oddness) and the sum identity for the tangent function, we get

$$\begin{aligned}\tan\alpha &= \tan\left(\pi - (\beta + \gamma)\right) \\ &= -\tan(\beta+\gamma) \\ &= \frac{-\tan\beta - \tan\gamma}{1 - \tan\beta\tan\gamma}.\end{aligned}$$

Hence

$$(1 - \tan\beta\tan\gamma)\tan\alpha = -\tan\beta - \tan\gamma.$$

By expanding the left-hand side and rearranging, we get the desired result. □

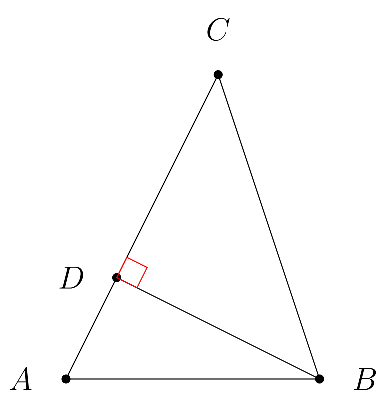
5 Trigonometric Functions in Geometry

Trigonometric functions have many useful applications in geometry. This is natural because our definition of trigonometric functions relied on the geometry of right triangles. Our first result in called the Law of Cosines, because it relates the length of the sides of any triangle to the cosine of one of its angles.

Theorem 5.1. *(Law of Cosines) Let ΔABC be a triangle with side lengths a, b, and c, and let the measure of the angle opposite the side with length c be denoted by C. Then, we have*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Proof. We create the following diagram.



Moreover, we have a triangle ΔABC (with $AB = c$, $BC = a$, and $AC = b$) and we just drew the altitude from B, intersecting $\overline{AC}$ at point D. Now, we see that $BD = a\sin C$ and $CD = a\cos C$. Furthermore, we have $AD = b - CD = b - a\cos C$. Applying the Pythagorean Theorem on ΔABD, we get

$$\begin{aligned}c^2 &= (b - a\cos C)^2 + (a\sin C)^2 \\ &= b^2 - 2ab\cos C + a^2\left(\sin^2 C + \cos^2 C\right) \\ &= a^2 + b^2 - 2ab\cos C,\end{aligned}$$

as desired. □

Next, we discuss a similar result call the Law of Sines. It relates the side lengths of a triangle to its angles.

Theorem 5.2. *Let ΔABC be a triangle with side lengths $BC = a$, $AC = b$, $AB = c$, and circumradius R. Then*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof. We use the same diagram as the proof of the law of cosines. For convenience, let $BD = h$. Then $\sin C = \frac{h}{a}$ and $\sin A = \frac{h}{c}$.

From this, we get that $\frac{c}{\sin C} = \frac{ca}{h}$ and $\frac{a}{\sin A} = \frac{ca}{h}$. Thus, $\frac{a}{\sin A} = \frac{c}{\sin C}$. Now, we can draw the altitudes from A and C and repeat the above calculations to get the respective equalities for the pairs of other sides. Now, we leave it as an exercise to the reader to involve the $2R$ term. One possible hint is to drop down the perpendiculars from the circumcenter to the sides of the triangle. □

Checkpoint 5.1. Let ΔABC be a triangle with $BC = a$ and circumradius R. Show that

$$\frac{a}{\sin A} = 2R.$$

From this proof, we can come up with another tidy result that can help us find the area of a triangle quickly.

Theorem 5.3. *Let ΔABC be a triangle with legs with length a, b, and c, and the angle in between them measuring C. Then the area of ΔABC is $\frac{1}{2}ab\sin C$.*

Proof. We draw the altitude from B, letting its height be h. Then $h = a \sin C$, so the area of the triangle is

$$[ABC] = \frac{1}{2}bh = \frac{1}{2}ab\sin C,$$

as desired. □

These are definitely a helpful tool to have when solving geometry problems, but they aren't essential. This is because these laws were simply derived by drawing the altitudes of a triangle, which anyways should be tried when stuck on a geometry problem. The exercises later in this section and chapter serve as a way to practice your trigonometry skills. Yes, you can solve these problems without trigonometry, but hone this skill so it can be used effectively and efficiently when you are taking the actual exam.

Now, you may be wondering how we find the values of sine and cosine evaluated at certain, commonly-occurring angles (shown in the below table). These values are derived by noting geometric relationships (as with 45-45-90 and 30-60-90 right triangles) and applying various trigonometric identities to these (e.g. double angle, half angle, and trigonometric addition/subtraction). These in some cases may take a bit of computation, so it's worthwhile to memorize the sine and cosine of certain angles. These angles will also be highlighted in the below table in bold. The values of sine and cosine at other common angles in this table and in the range $[0, 2\pi]$ can be found by applying the periodic and odd-even identities.

θ	$\sin\theta$	$\cos\theta$
0°	0	1
15°	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$
18°	$\frac{\sqrt{5}-1}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
36°	$\sqrt{\frac{5-\sqrt{5}}{8}}$	$\frac{\sqrt{5}+1}{4}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
72°	$\sqrt{\frac{5+\sqrt{5}}{8}}$	$\frac{\sqrt{5}-1}{4}$
75°	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
90°	1	0

6 Problem Solving in Trigonometry

In this final section, we'll illustrate some important problem solving skills in trigonometry through a few examples.

Example 6.1. At how many points do the graphs $y = \sin x$ and $y = \cos x$ intersect in the interval $[-4\pi, 4\pi]$?

Solution. One can simply plot both the sine and cosine functions and count the number of intersection points. One can observe that there are two intersection points in the interval $[0, 2\pi]$. We can use the fact that both sine and cosine have the same periods to get that since there are four of these periods in the interval $[-4\pi, 4\pi]$, our answer is $4 \cdot 2 = \boxed{8}$.

We can also approach the problem another way. This way may seem harder, but it should test your understanding of the graph and properties of the tangent function.

If the two functions intersect at a point with x-coordinate x, then we must have $\sin x = \cos x$. Now, we can't have $\sin x = \cos x = 0$, because that violates the Pythagorean trigonometric identity. Thus, we can divide both sides of $\sin x = \cos x$ be $\cos x$ to get $\tan x = 1$.

Since the range of $y = \tan x$ in each period is $[-\infty, \infty]$ and each period of $\tan x$ is strictly increasing and continuous, we can conclude that there exists exactly one solution x to the equation $\tan x = 1$ in every period of the tangent function. Now, how many periods are in the interval $[-4\pi, 4\pi]$? Since the period has length π, the number of solutions in the interval $\left[-\frac{(2k+1)\pi}{2}, \frac{(2k+1)\pi}{2}\right]$ is $2k+1$. So the number of solutions in the interval $\left[-\frac{7\pi}{2}, \frac{7\pi}{2}\right]$ is 7.

Now, we are left with the two intervals $\left[-4\pi, -\frac{7\pi}{2}\right]$ and $\left[\frac{7\pi}{2}, 4\pi\right]$; how many solutions are in these two pieces? Again from the graph of the tangent function, the one solution in each period is one the "right half" of the period. Since the first piece is a right half of a period but the second piece is not, we get one additional solution in these two intervals. Therefore, our answer is $7 + 1 = 8$. $\triangle$

The previous example illustrated the usefulness of graphing trigonometric functions in solving problems. The next example involves a more algebraic approach.

Example 6.2. Show that

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}.$$

Solution. First, notice that the left-hand side is in terms of tangents whereas the right-hand side is in terms of sines and cosines. So one natural way to proceed is to substitute each of the sine and cosine terms on the right-hand side by an equivalent expression in terms of tangents. How do we do this? One way to do this is as follows:

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ &= 2\cos^2\frac{x}{2}\tan\frac{x}{2} \\ &= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} \\ &= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}.\end{aligned}$$

We can also get the substitution

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

using similar techniques. We leave the details of deriving this identity as a later checkpoint.

You may be wondering why we started off by using the Half-angle formula. We did this because on the left-hand side of the desired equation, $\frac{\alpha+\beta}{2}$ is essentially the sum of two half angles.

We now plug in the substitutions into the right-hand side of the desired equation to get

$$\begin{aligned}
\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} &= \frac{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}} \\
&= \frac{2\tan\frac{\alpha}{2}\left(1+\tan^2\frac{\beta}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)+\left(1-\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\alpha}{2}\right)} \\
&= \frac{2\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{2\left(1-\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\right)} \\
&= \frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}} \\
&= \tan\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) \\
&= \tan\left(\frac{\alpha+\beta}{2}\right),
\end{aligned}$$

as desired. △

Checkpoint 6.1. Show that

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

Our final example is a geometry problem, that can be solved simply using basic trigonometric results.

Example 6.3. In triangle ABC, $AB = 4$, $BC = 6$, and $AC = 8$. Squares $ABQR$ and $BCST$ are drawn external to and lie in the same plane as ΔABC. Compute QT. *Source: ARML*

Solution. We aim to use the Law of Cosines on triangle ΔTBQ with respect to $\angle TBQ$. To do this, we need to find $\angle TBQ$, TB, and BQ. Notice that since $BCST$ and $ABQR$ are squares, $BQ = AB = 4$ and $TB = BC = 6$. Now, let $\angle TBQ = x$. By looking at the angles incident at vertex B, we get that $\angle CBA = 180 - x$. Applying the Law of Cosines to triangle ΔABC with respect to $\angle CBA$, we get

$$8^2 = 6^2 + 4^2 - 2\cdot 4\cdot 6\cos(180-x).$$

Solving for $\cos(180-x)$, we get $\cos(180-x) = -\frac{1}{4}$. Using the identity that $\cos(180-x) = -\cos x$, we get that $\cos x = \frac{1}{4}$ Now, we can apply the Law of Cosines on triangle ΔTBQ with respect to $\angle TBQ$ to get

$$QT^2 = 4^2 + 6^2 - 2\cdot 4\cdot 6\cdot\cos x.$$

Plugging in $\cos x = \frac{1}{4}$ and solving for QT, we get $QT = \boxed{2\sqrt{10}}$. △

7 Problems

1. $\star$ In triangle $\triangle JKL$, $\tan K = \frac{3}{2}$, angle J is right, and JK equals 2. What is KL? *Source: AoPS*

2. $\star$ Show that
$$\csc^2(x) + \sec^2(x) - \cot^2(x) = 2 \; + \tan^2(x)$$

3. $\star$ Let ΔABC have side lengths $AB = 5$, $BC = 12$, and $AC = 13$. Evaluate $\sin(\theta) * \cos(\theta) * \tan(\theta)$, where $\theta = \angle BAC$.

4. $\star$ Calculate $\sin(\frac{\pi}{12})$ and $\cos(\frac{\pi}{12})$.

5. $\star$ Show that $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$.

6. $\star$ Show that
$$\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y).$$

7. $\star$ Show that
$$\tan 2x = \frac{2\sin x}{\sec x\,(\cos^4 x - \sin^4 x)}.$$

8. $\star$ Show that
$$-\cot x = \frac{\sin 3x + \sin x}{\cos 3x - \cos x}.$$

9. $\star$ Show that
$$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}.$$

10. $\star$ Evaluate the expression
$$\frac{\tan 45^\circ \sec 74^\circ \tan 75^\circ \sin 26^\circ \cos 67^\circ}{\sin 23^\circ \cos 25^\circ \tan 40^\circ \tan 50^\circ \csc 65^\circ}$$

11. $\star$ What is $\sin 2x$ if $\csc^2 x + \sec^2 x = 10$?

12. $\star$ Triangle ABC has side lengths $AB = 5\sqrt{2}$, $BC = 9$, and $CA = 13$. What is the measure angle of ABC? *Source: AoPS*

13. $\star\star$ $ABCD$ is a regular tetrahedron (right pyramid whose faces are all equilateral triangles). If M is the midpoint of $\overline{CD}$, then what is $\cos\angle ABM$?

14. $\star\star$ What is the maximum value a, such that in a sequence $x_1, x_2, \ldots, x_a$ such that there are no two numbers c and d that satisfy $0 \le \frac{x_c - x_d}{1 + x_c x_d} \le 1$?

15. $\star\star$ Given that $x + y = 80^\circ$ and $\tan x \tan y + \tan x - \tan y + 1 = 0$, find x.

16. $\star\star$ Given that $(1 + \sin t)(1 + \cos t) = 5/4$, find the value of $(1 - \sin t)(1 - \cos t)$, Source: AIME, Ans: $\frac{13}{4} - \sqrt{10}$

17. ⋆⋆ Triangle ABC is isosceles, with $AB = AC$ and altitude $AM = 11$. Suppose that there is a point D on $\overline{AM}$ with $AD = 10$ and $\angle BDC = 3\angle BAC$. Find the perimeter of $\triangle ABC$ Source: AIME, Ans: $\sqrt{605} + 11$

18. ⋆⋆ Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

Source: AIME, Ans: 1

19. ⋆⋆ For what values is $\tan(3\theta) > 1$?

20. ⋆⋆ Show the following identities:

$$\arccos x \pm \arccos y = \arccos\left(xy \mp \sqrt{(1-x^2)(1-y^2)}\right) \tag{1}$$

$$\arccos x \pm \arccos y = \arccos\left(xy \mp \sqrt{(1-x^2)(1-y^2)}\right) \tag{2}$$

$$\arctan x \pm \arctan y = \arctan\left(\frac{x \pm y}{1 \mp xy}\right) \tag{3}$$

21. ⋆⋆ Triangle ABC has sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of length 43, 13, and 48, respectively. Let ω be the circle circumscribed around $\triangle ABC$ and let D be the intersection of ω and the perpendicular bisector of $\overline{AC}$ that is not on the same side of $\overline{AC}$ as B. The length of $\overline{AD}$ can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find the greatest integer less than or equal to $m + \sqrt{n}$. Ans, *Source: Mock AIME* Ans, 12

22. ⋆⋆ (Tangent half-angle substitution) For $x \in [-\pi, \pi]$, let $t = \tan(x/2)$. Show that

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}} \qquad \text{and} \qquad \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}.$$

23. Given $\tan\theta \sec\theta = 1$, find

$$\frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta}.$$

Source: AoPS

24. ⋆⋆ Find all solutions, in radians, to the equation $\cos 2x(1 + \tan 2x) = 1$.

25. ⋆⋆ What's the largest possible area of a triangle that has side lengths 6, 2,020, and x, where x is a real number, and why?

26. ⋆⋆ Find the number of solutions to

$$\sin x = \left(\frac{1}{2}\right)^x$$

on the interval $(0, 100\pi)$. *Source: AoPS*

27. ⋆⋆ Let S be the set of all real numbers x such that $0 \leq x \leq 2016\pi$ and $\sin x < 3\sin(x/3)$. The set S is the union of a finite number of disjoint intervals. Compute the total length of all these intervals. *Source: Math Prize For Girls* Ans: 1008π

28. ⋆⋆ Triangle ΔABC has a property where $\tan A = 2$ and $\tan B = 3$, find the angle value of C.

29. ⋆⋆ For any angle $0 < \theta < \frac{\pi}{2}$, show that

$$0 < \sin\theta + \cos\theta + \tan\theta + \cot\theta - \sec\theta - \csc\theta < 1.$$

Source: HMMT

30. ⋆⋆⋆ Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

31. ⋆⋆⋆ Triangle ΔABC has side lengths AB = 3, BC = 4, and AC = 5. A crease is folded such that point A maps to A'. A' lies on BC, such that A'B = 1. What is the length of the crease?

32. ⋆⋆⋆ Let $f(x) = \log_{10}(\sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x))$. The intersection of the domain of $f(x)$ with the interval $[0, 1]$ is a union of n disjoint open intervals. What is n? *Source: AoPS*

33. ⋆⋆⋆ Find the number of solutions to the equation

$$\tan(5\pi\cos\theta) = \cot(5\pi\sin\theta)$$

where $\theta \in (0, 2\pi)$. *Source: AoPS*

34. ⋆⋆⋆ Let S be the set of all real values of x with $0 < x < \frac{\pi}{2}$ such that $\sin x$, $\cos x$, and $\tan x$ form the side lengths (in some order) of a right triangle. Compute the sum of $\tan^2 x$ over all x in S. *Source: Math Prize For Girls* Ans: $\sqrt{2}$

35. ⋆⋆⋆ Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$. *Source: AIME* Ans: 108

36. ⋆⋆⋆ If $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$, compute x^2. *Source: AoPS*

37. ⋆⋆⋆ If

$$\frac{\sin x}{\cos y} + \frac{\sin y}{\cos x} = 1 \quad \text{and} \quad \frac{\cos x}{\sin y} + \frac{\cos y}{\sin x} = 6,$$

then find $\frac{\tan x}{\tan y} + \frac{\tan y}{\tan x}$. *Source: AoPS*

38. $\star\star\star$ A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$. *Source: AIME* Ans, 183

39. $\star\star\star$ Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$, and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120°$. *Source: BMO*

40. $\star\star\star$ Let α and β be angles for which

$$\frac{\sec^4 \alpha}{\tan^2 \beta} + \frac{\sec^4 \beta}{\tan^2 \alpha}$$

is defined. Find the minimum value of the expression. *Source: AoPS*