Iowa City Math Tournament

Iowa City Math Circle

August 9, 2020

1 Rules 🔒

- There will be two rounds in the written portion of the tournament: the Speed Round and the Focus Round. The Speed Round has **20 questions** and lasts **30 minutes**, testing competitors agility in solving easier problems. In contrast, the Focus Round is a **1 hour** test that has **10 questions**, challenging competitors to use their problem-solving skills and creativity to solve harder problems.

- Any aids besides blank scratch paper and writing utensils are not allowed during either written round. Therefore, **calculators, rulers, compasses, and graph paper are not allowed**.

- Competitors should time each round themselves, and should not work overtime. You are not allowed to work on one written portion in the time frame of the other.

- Competitors should submit their answers to the Google form (https://docs.google.com/forms/d/e/1FAIpQLSdQ-qhoh-WezqsJQULXMa_B4ArqNTn_xQDy2Zt1AATFscJ5wg/viewform) once they've finished both the written rounds. All answers should be integers, so formatting your answers in the form shouldn't be an issue. Just make sure to not add any extra spaces or units; all the answers will be graded by a computer. We recommend taking the test on paper, then copying answers over into the form afterwards.

- Students earn one point for each correct answer on the Speed Round and three points for each correct answer on the Focus Round. There is no penalty for incorrect answers. A competitor's individual score is the sum of the number of points earned in the Speed and Focus Rounds.

- If you believe a problem needs clarification or you believe an answer is graded incorrectly please do not hesitate to contact us by email:
iowacitymathcircle@gmail.com

2 Speed Round

Time Limit: **30 minutes**

1. A number is tripled and then added to 80 to produce 120. The number can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. Mr. Norton is writing problems for the next math club meeting. He was able to write 5 problems in 18 minutes. Given that he continues to write at the same rate, how many minutes will it take for him to write 10 more problems?

3. If $9a + 5b + 2c = 10$ and $a + 3c = 15$, what is $2a + b + c$?

4. James has a two liter bottle of pop. Before drinking it, the bottle was 80% full. After drinking some pop, the bottle became 1/3 full. The number of liters James drank can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. Xiang flips a fair coin 2020 times. What is the probability that he will flip one more heads than tails, rounded to the nearest percent? For example, if you computed the probability to be $\frac{2}{3}$, your answer would be 67.

6. There is a set containing 7 numbers that have an mean of 20. A number is added to the set, and the new average of these 8 numbers is 26. What is the newly added number?

7. A cafe offers a build your own sandwich station. There, you could configure your sandwich with 2 types of bread, 3 types of cheese, and any combination of 6 different types of vegetables (you can load none or all of them, for example). How many total combinations of sandwiches exist with exactly 1 type of bread, 1 type of cheese, and any combination of vegetables?e

8. If the sum of two numbers equals 20 and their product equals 17, what is the sum of the squares of these two numbers?

9. Let a be a positive integer, and let n be the sum of the first a positive integers. What is the least possible value of a that makes n divisible by 16?

10. Let $ABCD$ be a quadrilateral with $AB = AD = 5$, $BC = CD = 12$, and $\angle ABC = 90°$. The length of diagonal $\overline{BD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. Find base b if $72_b + 153_b = 245_b$ (the subscript b indicates that the numerals are in base b).

12. There are 2 English textbooks, 3 math textbooks, and 4 science textbooks on a shelf. All the textbooks are distinct. Sayaka takes three textbooks from her shelf by random. The probability that exactly two of the three textbooks are of the same subject can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. John has 7 black shirts and 3 white shirts. On any given day, he picks a random shirt and wears it. After wearing it for the day, he doesn't wear the same shirt again. The probability that he picks at least 2 black shirts in the first 3 days can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

14. Let $ABCD$ be an isosceles trapezoid with $AD = BC$. Given that $AC = 4$, $AD = 3$, and $\angle CAD = 90°$, find $AB \cdot CD$.

15. A 2 row by 3 column grid of squares exists on a plane. Each square is colored red or blue, by random. The probability that at least one of the rows has all squares colored the same color can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

16. Akemi wants to buy a piece of candy for 40 cents from the grocery store. Her only means of payment to the cashier are coins, having an unlimited supply of quarters, dimes, nickels, and pennies. Assuming that she pays the 40 cents exactly, how many ways can she pay the 40 cents with her unlimited supply of coins? The order in which she presents the coins to the cashier is irrelevant.

17. Let ABC be a triangle with $AB = 9$, $BC = 15$, and $AC = 16$. In addition, let point D be the point in the interior of the triangle that lies on both the median of side $\overline{AB}$ (i.e. the segment that passes through point C and the midpoint of $\overline{AB}$) and the angle bisector of angle $\angle BAC$. Let E be the point on side $\overline{BC}$ that lies on line $\overline{AD}$. Finally, extend $\overline{BD}$ to point F on side $\overline{AC}$. The ratio $\frac{CF}{BE}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

18. How many ways can you choose 3 subsets A_1, A_2, and A_3 from the set $\{1,2,3\}$ such that A_1 is a proper subset of A_2 and A_2 is a subset of A_3? A proper subset of a set S is a subset of S that is not equal to S.

19. Two real numbers are randomly chosen exclusively between 0 and 1.9. What is the probability that the sum of the two numbers is greater then their product, rounded to the nearest percent?

20. Reece and Kevin play the following game: each roll a blue die and a red die. Each of their scores is the value of the blue die divided by the value of the red die. The probability that Reece obtains a greater score than Kevin can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3 Focus Round ⚙

Time Limit: **1 hour**

1. Find the number of 11-digit positive integers such that every three consecutive digits form a palindrome. A palindrome is a number that reads the same forwards and backwards.

2. Let f be a function over the positive integers so that $f(k)$ equals the sum of the digits of $\underbrace{111\ldots1}_{\text{k 1's}}{}^2$, where there are k 1's in the number being squared. For example, $f(1) = 1$, $f(2) = 4$ (since the sum of the digits of 11^2 is 4), and $f(3) = 1 + 2 + 3 + 2 + 1 = 9$. Find $f(1) + f(2) + f(3) + \cdots + f(9)$.

3. Find the last two digits of the number n given by
$$n = 2021^{2021} + 1021^{1021} + 2021^{1021} + 1021^{2021}.$$

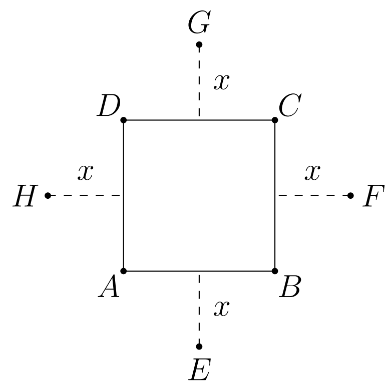
4. Let points A, B, C, and D lie on a circle (in that order), and let E be the intersection of $\overline{AC}$ and $\overline{BD}$. Given that $DE = 6$, $AE = 3$, $EC = 10$, and $\angle CBD = 90°$, find the radius of the circle.

5. Let $ABCD$ be an isosceles trapezoid with $AB = 2$, $CD = 4$, $AD = BC$, and $\angle ABC = 135°$. Let V b the volume of the solid created when $ABCD$ is rotated 360° along side BC. V can be written in the form $\frac{a\pi\sqrt{b}}{c}$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a + b + c$.

6. Let $A_1B_1C_1$ be a triangle with interior angles measuring $\angle A_1 = 40°$, $\angle B = 60°$, and $\angle C = 80°$, and let circle O be the circumcircle of $\Delta A_1B_1C_1$. Let A_2 be the point where the angle bisector of interior angle $\angle A_1$ intersects circle O, B_2 be the point where the angle bisector of interior angle $\angle B_1$ intersects circle O, and C_2 be the point where the angle bisector of interior angle $\angle C_1$ intersects circle O. Now, consider triangle $A_2B_2C_2$ and the three points formed by intersecting its three interior angle bisectors with circle O. Call these three points A_3, B_3, and C_3. Similarly, points A_4, B_4, and C_4 are formed by the angle bisectors of $\Delta A_3B_3C_3$ intersecting the circle. This process is repeated until we obtain the points A_{10}, B_{10}, and C_{10}. Find the the measure, in degrees, of the middle (not the largest nor the smallest) of the three interior angles of $\Delta A_{10}B_{10}C_{10}$.

7. An Energizer bunny is placed on the coordinate plane at $(0, 0)$. A move with length x is defined as the bunny moving strictly north, east, south, or west x units. A bunny makes four moves with length 1, 2, 4, and 8 in that order. Find the number of possible coordinate locations the bunny can end up after these four moves.

8. Let $ABCD$ be a square with side length 2. Construct points E, F, G, and H (outside the square) on the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively such that the distance from each of these points to the respective sides is x units, as shown in the figure below. Let S be the sum of all finite, positive values of x so that the region of overlap between the triangles ΔABG, ΔBCH, ΔCDE, and ΔDAF is a regular polygon. S can be expressed as $\sqrt{a} - b$, where a and b are positive integers and a is not divisible by the square of any prime. Find $a - b$.



9. For all real y, the expression $x^3 + 2x^2 + 2yx^2 - 4x + xy^2 + 8 - y^2$ is greater than or equal to 0 for all $x \geq c$. Find the minimum possible value of c.

10. Given $ab + bc + ca = 2$, find the sum of all possible values of $a + b + c$ if a, b, and c are real numbers satisfying

$$a^3 - a^2 + b^3 - b^2 + c^3 - c^2 = ab(3c + 2) + 2c(a + b).$$

—— **End of Written Test** ——