5 Answers 🔑

5.1 Speed Round

1. 43
2. 36
3. 5
4. 29
5. 0%
6. 68
7. 384
8. 366
9. 31
10. 133
11. 8
12. 139
13. 109
14. 7
15. 23
16. 31
17. 391
18. 18
19. 100%
20. 1901

5.2 Focus Round

1. 90
2. 285
3. 84
4. 7
5. 19
6. 60
7. 256
8. 1
9. 2
10. 3

5.3 Countdown Round

1. 1
2. 5 free throws
3. 45 ways
4. 216
5. 0
6. $25,000$
7. 260
8. $\frac{1}{2}$
9. $\frac{3\sqrt{3}}{4}$
10. 1
11. 21
12. 0 pages
13. 10 bags
14. 5
15. 49

16. 9
17. 2500
18. 6
19. 2
20. 24
21. -10
22. 108
23. $\frac{3}{5}$
24. 180 ways
25. 119°
26. 60
27. $\frac{2}{5}$
28. 56
29. 4 moves
30. 7th place
31. 18
32. 36 triangles
33. 202
34. $\frac{12}{25}$
35. $3\sqrt{3}$
36. $\frac{10}{13}$
37. 1
38. 40
39. -50
40. 19 pens
41. 100π
42. 12.5

43. $2,500$ hours
44. Saturday
45. $6\sqrt{3}$
46. $75,000$ words per day
47. Tuesday
48. $\frac{1}{7}$
49. 24 hours
50. $\frac{21}{50}$

6 Solutions

6.1 Speed Round

1. A number is tripled and then added to 80 to produce 120. The number can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Solution. Let n be the number. Then we have the equation $3n + 80 = 120 \implies n = \frac{40}{3}$. So the answer is $40 + 3 = \boxed{43}$. △

2. Mr. Norton is writing problems for the next math club meeting. He was able to write 5 problems in 18 minutes. Given that he continues to write at the same rate, how many minutes will it take for him to write 10 more problems?

   Solution. Writing 10 problems will take twice as much time as writing 5 problems, so it will take $2 \cdot 18 = \boxed{36}$ minutes for Mr. Norton to write 10 more problems. △

3. If $9a + 5b + 2c = 10$ and $a + 3c = 15$, what is $2a + b + c$?

   Solution. Adding the two equations together, we get $10a+5b+5c = 25$. Dividing by 5, we get $2a + b + c = \boxed{5}$. △

4. James has a two liter bottle of pop. Before drinking it, the bottle was 80% full. After drinking some pop, the bottle became 1/3 full. The number of liters James drank can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Solution. $80\% = \frac{4}{5}$. James drank $\frac{4}{5} - \frac{1}{3} = \frac{7}{15}$ of the bottle, so James drank $\frac{7}{15} \cdot 2 = \frac{14}{15}$ liters. So the answer is $14 + 15 = \boxed{29}$. △

5. Xiang flips a fair coin 2020 times. What is the probability that he will flip one more heads than tails, rounded to the nearest percent? For example, if you computed the probability to be $\frac{2}{3}$, your answer would be 67.

   Solution. Let the number of tails Xiang flips be n, and suppose that Xiang flips $n+1$ heads. Then Xiang flips the coin $n+n+1 = 2n+1$ times in total. However, $2n + 1$ is always odd, while 2020 is even, so Xiang can never flip one more heads than tails. Thus, the probability is $\boxed{0}$%. △

6. There is a set containing 7 numbers that have an mean of 20. A number is added to the set, and the new average of these 8 numbers is 26. What is the newly added number?

   Solution. The sum of the original 7 numbers is $7 \cdot 20 = 140$. The sum of the 8 numbers is $8 \cdot 26 = 208$. Thus, the newly added number must be $208 - 140 = \boxed{68}$. △

7. A cafe offers a build your own sandwich station. There, you could configure your sandwich with 2 types of bread, 3 types of cheese, and any combination of 6 different types of vegetables (you can load none or all of them, for example). How many total combinations of sandwiches exist with exactly 1 type of bread, 1 type of cheese, and any combination of vegetables?

   Solution. Each type of vegetable can either be in the sandwich or not in the sandwich. Therefore, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$ possible combinations of vegetables. Then overall, there will be $2 \cdot 3 \cdot 64 = \boxed{384}$ sandwich combinations. △

8. If the sum of two numbers equals 20 and their product equals 17, what is the sum of the squares of these two numbers?

   Solution. Let the two numbers be a and b. Then $a + b = 20 \implies (a+b)^2 = 400 \implies a^2 + 2ab + b^2 = 400$. Since we are given $ab = 17$, we subtract $2ab = 34$ from the previous equation to get $a^2 + b^2 = 400 - 34 = \boxed{366}$. △

9. Let a be a positive integer, and let n be the sum of the first a positive integers. What is the least possible value of a that makes n divisible by 16?

   Solution. We could solve this problem by listing n on the first numbers of a until we could found the answer, but we will present a more clever solution. Since n is the sum of the first a integers, we can write n as $\frac{a(a+1)}{2}$, which mus be divisible by 16. Therefore, $a(a+1)$ must be divisible by $16 \cdot 2 = 32 = 2^5$. Note that both a and $a+1$ cannot be divisible by 2, hence, 2^5 must divide either a or $a+1$. The smallest value of a such that $32|a$ is $a = 32$, and the smallest value of a such that $32|a+1$ is $a = 31$. Hence, the least value of a such that n is divisible by 16 is $a = \boxed{31}$. △

10. Let $ABCD$ be a quadrilateral with $AB = AD = 5$, $BC = CD = 12$, and $\angle ABC = 90°$. The length of diagonal $\overline{BD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

    Solution. Notice that triangle ABC is a 5-12-13 right triangle, so $AC = 13$. Since $ABCD$ is a kite, its diagonals will be perpendicular. This means that BD is double the altitude of triangle ABC with respect to AC.

    Let h be the altitude of triangle ABC with respect to AC. The area of ABC is $\frac{1}{2} \cdot 5 \cdot 12 = 30$. We can then use the hypotenuse to find h, by finding the area that way. This gets the equation $\frac{1}{2} \cdot 13h = 30$. Thus $h = \frac{60}{13}$, and consequently $BD = 2 \cdot \frac{60}{13} = \frac{120}{13}$. So the answer is $120 + 13 = \boxed{133}$. △

11. Find base b if $72_b + 153_b = 245_b$ (the subscript b indicates that the numerals are in base b).

Solution. We first rewrite the equation in base-10: $7b+2+b^2+5b+3 = 2b^2+4b+5$. Simplifying gives us the quadratic $b^2 - 8b = 0$, which has two solutions: $b = 0$ and $b = 8$. Since 0 is not a valid base, $b = \boxed{8}$. △

12. There are 2 English textbooks, 3 math textbooks, and 4 science textbooks on a shelf. All the textbooks are distinct. Sayaka takes three textbooks from her shelf by random. The probability that exactly two of the three textbooks are of the same subject can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

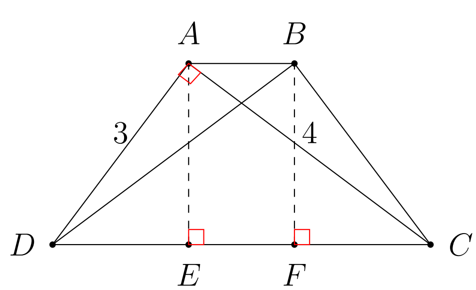
Solution. There are 9 textbooks in total on the shelf, so there are $\binom{9}{3} = 84$ total ways to choose three textbooks from the shelf. Sayaka can pick exactly two English textbooks in $\binom{2}{2} \cdot 7 = 7$ ways, exactly two math textbooks in $\binom{3}{2} \cdot 6 = 18$ ways, and exactly two science textbooks in $\binom{4}{2} \cdot 5 = 30$ ways. Thus the probability that she chooses exactly two textbooks of the same subject will be $\frac{7+18+30}{84} = \frac{55}{84}$. So the answer is $55 + 84 = \boxed{139}$. △

13. John has 7 black shirts and 3 white shirts. On any given day, he picks a random shirt and wears it. After wearing it for the day, he doesn't wear the same shirt again. The probability that he picks at least 2 black shirts in the first 3 days can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. First, the probability that John picks three black shirts in the first three days will be $\frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} = \frac{210}{720}$. Next, suppose John picks two black shirts and one white shirt. There are three possible orders in which he can do so. The numerator and denominator of the probability for each order will be the same: $\frac{7 \cdot 6 \cdot 3}{10 \cdot 9 \cdot 8} = \frac{126}{720}$. Thus, the total possibility that John picks at least 3 black shirts will be $\frac{210+3 \cdot 126}{720} = \frac{49}{60}$. So the answer is $49 + 60 = \boxed{109}$. △

14. Let $ABCD$ be an isosceles trapezoid with $AD = BC$. Given that $AC = 4$, $AD = 3$, and $\angle CAD = 90^\circ$, find $AB \cdot CD$.

Solution. We draw the following figure, in which we down the altitudes of the trapezoid from points A and B (intersecting the base $\overline{CD}$ at points E and F) and draw in the diagonals.



From the Pythagorean theorem, we have that $CD = 5$. Furthermore, since $\overline{AE}$ is an altitude of right triangle ΔCDA, it's length is $\frac{3\cdot 4}{5} = \frac{12}{5}$. Now by applying the Pythagorean theorem to ΔADE (or by noting that this triangle is similar to ΔCDA) we get that $DE = \frac{9}{5}$. By symmetry, $CF = \frac{9}{5}$ as well. Using this, we have $EF = CD - DE - CF = 5 - 2 \cdot \frac{9}{5} = \frac{7}{5}$. Since $ABFE$ is a rectangle by construction, $AB = EF = \frac{7}{5}$. Hence, our answer is $\frac{7}{5} \cdot 5 = \boxed{7}$. △

15. A 2 row by 3 column grid of squares exists on a plane. Each square is colored red or blue, by random. The probability that at least one of the rows has all squares colored the same color can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Instead of trying to compute the probability that at least one of the rows have all the same color, we should compute the probability that none of the rows have the same color. Each row can be colored 8 ways (3 tiles with 2 colors each), and 2 of the ways will have all of the tiles in the same color (RRR and BBB). Therefore, we have a $\frac{1}{4}$ chance that the tiles in a row have the same color, so there is a $\frac{3}{4}$ chance that a row has tiles with different colors. Since we have two different rows, there is a $(\frac{3}{4})^2$ or $\frac{9}{16}$ chance that all rows will have different colors. Now, we can take the complement of that to find the chance that at least one of the rows have a different color, which is $1 - \frac{9}{16} = \frac{7}{16}$. So the answer is $7 + 16 = \boxed{23}$. △

16. Akemi wants to buy a piece of candy for 40 cents from the grocery store. Her only means of payment to the cashier are coins, having an unlimited supply of quarters, dimes, nickels, and pennies. Assuming that she pays the 40 cents exactly, how many ways can she pay the 40 cents with her unlimited supply of coins? The order in which she presents the coins to the cashier is irrelevant.

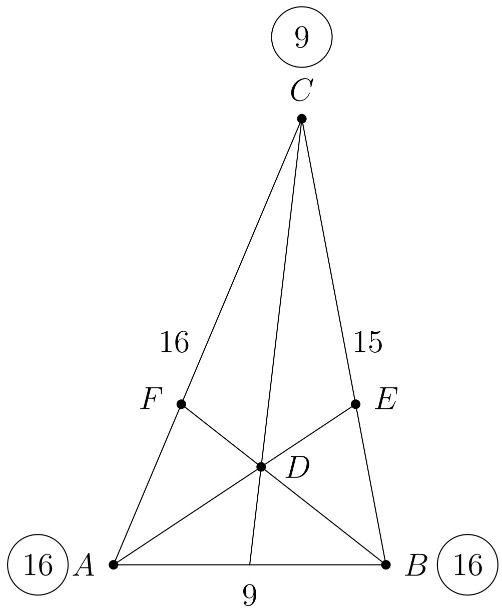
Solution. Let q, d, n, and p denote the number of quarters, dimes, nickels, and pennies, respectively, she uses to pay the 40 cents. The following table gives us all the possibilities we can pay the amount. This table can be constructed by systematically taking cases on the most expensive denomination used and how many of them are used. Also, by fixing q, d, and n, we can find p, so the problem can be reduced to finding all combinations of quarters, dimes, and nickels that total at most 40 cents. This means that we can ignore the p column, which you should do on the actual test; we just include it here for completeness. In the following table, we take cases on q and d, and then easily determine the number of values of n that can work.

q	d	n	p
1	1	1	0
1	1	0	5
1	0	3	0
1	0	2	5
1	0	1	10
1	0	0	15
0	4	0	0
0	3	2	0
0	3	1	5
0	3	0	10
0	2	4	0
0	2	3	5
0	2	2	10
0	2	1	15
0	2	0	20
0	1	6	0
0	1	5	5
0	1	4	10
0	1	3	15
0	1	2	20
0	1	1	25
0	1	0	30
0	0	8	0
0	0	7	5
0	0	6	10
0	0	5	15
0	0	4	20
0	0	3	25
0	0	2	30
0	0	1	35
0	0	0	40

Adding everything up, we have a total of $\boxed{31}$ ways. △

17. Let ABC be a triangle with $AB = 9$, $BC = 15$, and $AC = 16$. In addition, let point D be the point in the interior of the triangle that lies on both the median of side $\overline{AB}$ (i.e. the segment that passes through point C and the midpoint of $\overline{AB}$) and the angle bisector of angle $\angle BAC$. Let E be the point on side $\overline{BC}$ that lies on line $\overline{AD}$. Finally, extend $\overline{BD}$ to point F on side $\overline{AC}$. The ratio $\frac{CF}{BE}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. We'll use the following figure in the solution, which shows the masses of certain vertices.



We can use mass points to solve this problem as we aim to find ratios involving cevians. From the Angle-bisector theorem, $\frac{BE}{CE} = \frac{9}{16}$. Using this, we set the mass on B to be 16 and the mass on C to be 9. Since we know $BE + CE = 15$, we can calculate $BE = \frac{27}{5}$. By mass points, we get that the mass on A is 16 (the same mass as B) because the cevian through C bisects $\overline{AB}$. Now, we see that $\frac{CF}{AF} = \frac{16}{9}$. Since $CF + AF = 16$, we can calculate $CF = \frac{256}{25}$. Hence, our answer is $\frac{256}{25} \div \frac{27}{5} = \frac{256}{135}$. So the answer is $256 + 135 = \boxed{391}$. △

18. How many ways can you choose 3 subsets A_1, A_2, and A_3 from the set $\{1, 2, 3\}$ such that A_1 is a proper subset of A_2 and A_2 is a subset of A_3? A proper subset of a set S is a subset of S that is not equal to S.

Solution. Let's divide this problem up into cases where the cases are different sizes of the sets. In order to have a proper subset, the size of A_1 must be smaller than the size of A_2, and the size of A_2 must be smaller then the size of A_3. Considering this, we have four cases.

Case 1: A_3 has size 3, A_2 has size 2, and A_1 has size 1.

Here, we only have one option for A_3, which is $1, 2, 3$. From here, we have 3 options to pick which element will be omitted from A_1. Then, we have two options to pick which element will be omitted from A_2 to make A_1. Therefore, we have $3 \cdot 2 = 6$ total ways in this case.

Case 2: A_3 has size 3, A_2 has size 2, and A_1 has size 0.

This is similar to case 1, except that we only have 1 option when picking A_1, which is to pick the empty set. We have 3 total ways in this case.

Case 3: A_3 has size 3, A_2 has size 1, and A_1 has size 0.

Here, we have 3 options to pick the one element in A_2, and other then that our options are fixed. We have 3 total ways in this case.

Case 4: A_3 has size 2, A_2 has size 1, and A_1 has size 0.

In this case, we have 3 ways of choosing A_3, since we choose two out of three elements. Then, we have 2 ways of choosing the one element in A_2, and A_1 is fixed. Therefore, we have 6 total ways in this case.

Summing up the four cases gives us $\boxed{18}$ total ways. △

19. Two real numbers are randomly chosen exclusively between 0 and 1.9. What is the probability that the sum of the two numbers is greater then their product, rounded to the nearest percent?

Solution. Let the two numbers be a and b. We can graph the inequality $a+b > ab$ on the ab-axis. To do so, we'll solve $a + b = ab$ for b:

$$\begin{aligned} ab - b &= a \\ b(a-1) &= a \\ b &= \frac{a}{a-1} \\ b &= \frac{a-1+1}{a-1} \\ b &= 1 + \frac{1}{a-1} \end{aligned}$$

Note that $b = 1 + \frac{1}{a-1}$ is the graph of $b = \frac{1}{a}$ translated 1 unit up and 1 unit to the right. This is shown below:

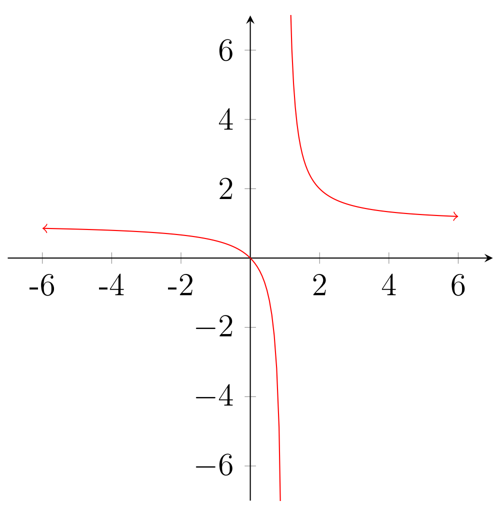


Figure 1: The figure shows the plots of $y = 1 + \frac{1}{x-1}$ on the interval $[-6, 6]$.

We can find the vertices of this hyperbola. Since the graph is symmetric along the $a = b$ line, the vertices occur when $a = b$. Thus, we plug $a = b$ into our original equation to get $b + b = b^2 \implies b^2 - 2b = 0$. Solving, we get that the locations of two vertices will be at $(0, 0)$ and $(2, 2)$, which are both conveniently outside of the bounds of the problem.

Lastly, we must check whether the area inside the bounds satisfies the inequality. To do so, we plug in a point in side the bounds, e.g. $(1, 1)$, into the inequality: $1 + 1 > 1 \cdot 1$. Since this is true, the entire area inside the bounds satisfies the inequality. So the probability that $a + b > ab$ will be $\boxed{100}$%. △

20. Reece and Kevin play the following game: each roll a blue die and a red die. Each of their scores is the value of the blue die divided by the value of the red die. The probability that Reece obtains a greater score than Kevin can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Suppose that if a player obtains a higher score than the other player, they "win" the game.

There are three scenarios: Reece wins, Kevin wins, or Reece ties with Kevin. By symmetry, the probability Reece wins is the same as the probability Kevin wins (as the ways in which they obtain their scores are identical). Let the probability that Reece wins be p, and the probability that they tie be q. Then we have $1 = p + p + q = 2p + q$ by summing up the probabilities of each of the three scenarios. Rearranging, we obtain $p = \frac{1-q}{2}$. So we can find p by calculating q, which seems easier to compute!

Be careful to notice that Reece and Kevin may not obtain the same values on their die, but can still tie! For example, Reece can obtain 1 on his blue die and a 2 on his red die, while Kevin can roll a 3 on his blue die and a 6 on his red die. They both get a score of $\frac{1}{2}$. With this in mind, let us find all groups of fractions that give us an equal score. We take cases by denominator (going sequentially from 1 to 6), and within each denominator, increasing the numerator from 1 through 6.

(a) Score $= \frac{1}{1}$. Possible fractions are $\frac{1}{1}, \frac{2}{2}, \ldots \frac{6}{6}$.
(b) Score $= \frac{2}{1}$. Possible fractions are $\frac{2}{1}, \frac{4}{2}, \frac{6}{3}$
(c) Score $= \frac{3}{1}$. Possible fractions are $\frac{3}{1}, \frac{6}{2}$.
(d) Score $= \frac{1}{2}$. Possible fractions are $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}$
(e) Score $= \frac{3}{2}$. Possible fractions are $\frac{3}{2}, \frac{6}{4}$
(f) Score $= \frac{1}{3}$. Possible fractions are $\frac{1}{3}, \frac{2}{6}$.
(g) Score $= \frac{2}{3}$. Possible fractions are $\frac{2}{3}, \frac{4}{6}$

If Reece and Kevin were to obtain the same score, then they must obtain two fractions in the same group, or obtain the same exact fraction (i.e. their rolls are identical).

(a) Case 1: Reece and Kevin have identical rolls. The probability that their blue die are equal is $\frac{1}{6}$, and the probability that their red die are equal is $\frac{1}{6}$. Therefore, the probability that they have identical rolls is $\frac{1}{36}$.

(b) Case 2: Reece and Kevin pick different fractions from the same group. Let a group have size n. Then Reece has n choices for his fraction, while Kevin has $n-1$ (because they cannot choose the same one, because this was covered in Case 1). Since each one has 36 total ways of rolling their pair of die, the probability is $\frac{n(n-1)}{36^2}$. Now we must sum these values over all groups. Our group sizes are $6, 3, 2, 3, 2, 2, 2$, giving us a total of $\frac{50}{36^2}$.

Therefore, the probability they tie is $\frac{1}{36} + \frac{50}{36^2} = \frac{43}{648}$, hence $p = \frac{1-\frac{43}{648}}{2} = \frac{605}{1296}$. So the answer is $605 + 1296 = \boxed{1901}$. △

6.2 Focus Round

1. Find the number of 11-digit positive integers such that every three consecutive digits form a palindrome. A palindrome is a number that reads the same forwards and backwards.

Solution. Let a and b be the first two digits of the 11-digit integer. Then in order for the first three digits of this number to be a palindrome, the third digit must be a. In order for the next set of three consecutive digits to be a palindrome, the fourth digit must be a b. By keep applying the constraint to every three consecutive digits, we see that the number is of the form $\underline{ababab \cdots aba}$, where all the odd-positioned digits are a and the even positioned digits are b. Hence, every valid 11-digit integer is determined uniquely by the choice of a and b. We must have that $a \neq 0$ (as the first digit of the number cannot be zero). Therefore, a can be 9 possible values $(1, 2, \ldots, 9)$ and b can be 10 possible values $(0, 1, \ldots, 9)$, so there are a total of $9 \cdot 10 = \boxed{90}$ valid 11-digit integers. △

2. Let f be a function over the positive integers so that $f(k)$ equals the sum of the digits of $\underbrace{111\ldots1}_{\text{k 1's}}{}^2$, where there are k 1's in the number being squared. For example, $f(1) = 1$, $f(2) = 4$ (since the sum of the digits of 11^2 is 4), and $f(3) = 1+2+3+2+1 = 9$. Find $f(1) + f(2) + f(3) + \cdots + f(9)$.

Solution. Let's see if we can notice a pattern:

$$1^2 = 1, 11^2 = 121, 111^2 = 12321, 1111^2 = 1234321, \text{etc.}$$

Therefore, we can use the above pattern to predict $111111111^2 = 12345678987654321$. Now, let's find the sum of the digits of these:

$$f(1) = 1, f(2) = 4, f(3) = 9, f(4) = 16.$$

Noticing the pattern that these are simply perfect squares, we can hypothesize $f(k) = k^2$. Why is this true? We see that

$$\begin{aligned} f(k) &= 1 + 2 + \cdots + k + (k-1) + \cdots + 2 + 1 \\ &= 2(1 + 2 + \cdots + k) - k \\ &= k(k+1) - k \\ &= k^2. \end{aligned}$$

Using the well-know fact that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ gives us that

$$\begin{aligned} f(1) + f(2) + f(3) + \cdots + f(9) &= 1^2 + 2^2 + \cdots + 9^2 \\ &= \frac{9(10)(19)}{6} \\ &= \boxed{285}. \end{aligned}$$

△

3. Find the last two digits of the number n given by

$$n = 2021^{2021} + 1021^{1021} + 2021^{1021} + 1021^{2021}.$$

Solution. Using Euler's theorem, we see that

$$2021^{2021} \equiv 1021^{1021} \equiv 2021^{1021} \equiv 1021^{2021} \equiv 21^{21} \pmod{100},$$

since each of the exponents of the four terms leave a remainder of 21 when divided by $\phi(100) = 40$ and each of the bases leave a remainder of 21 when divided by 100. Therefore, it suffices to find the last two digits of 21^{21}. To do this, we use the binomial expansion $(20 + 1)^{21}$, and can conveniently ignore all but 2 terms (i.e. those with 20 raised to the first and zero-th powers, as all greater powers would make the term divisible by 100). We have that

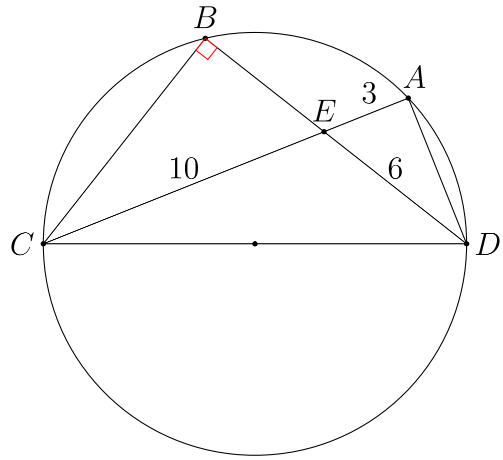
$$21^{21} \equiv (20 + 21)^{21} \equiv 21 \cdot 20 + 1 \equiv 21 \pmod{100}.$$

Since each of the four terms has last two digits 21, the last two digits of n are $\boxed{84}$.

△

4. Let points A, B, C, and D lie on a circle (in that order), and let E be the intersection of $\overline{AC}$ and $\overline{BD}$. Given that $DE = 6$, $AE = 3$, $EC = 10$, and $\angle CBD = 90°$, find the radius of the circle.

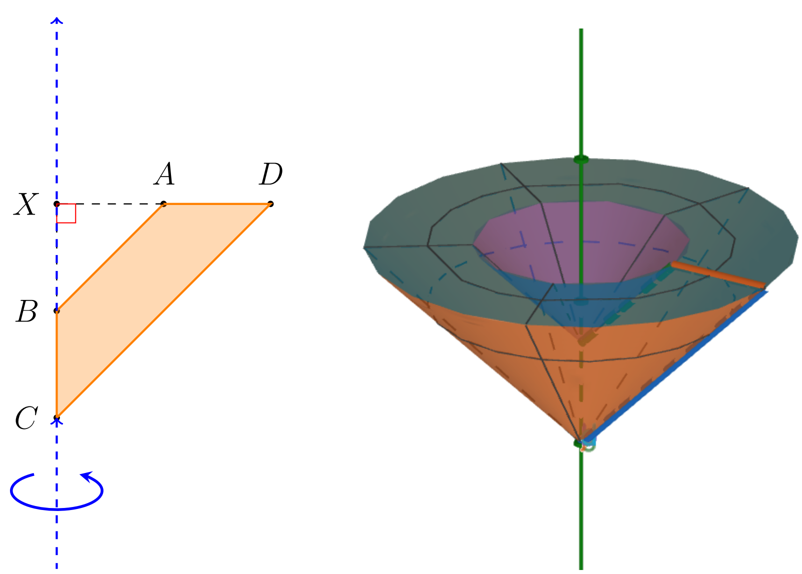
Solution. We draw the following figure.



Since C, B, and D are on the circle, and $\angle CBD = 90°$, CD is the diameter of the circle. This also means $\angle CAD = 90°$ because A is also on the circle. Now, lets take a look at right triangle ΔDAE. Using the Pythagorean theorem on that triangle (or ny noticing that it's a 30-60-90 triangle), we find that $DA = 3\sqrt{3}$. Using the Pythagorean theorem again on triangle ΔDAC, we find that $DC = 14$. Therefore, the radius of the circle is $\boxed{7}$ (note that finding EB is not needed to find the answer). △

5. Let $ABCD$ be an isosceles trapezoid with $AB = 2$, $CD = 4$, $AD = BC$, and $\angle ABC = 135°$. Let V b the volume of the solid created when $ABCD$ is rotated $360°$ along side BC. V can be written in the form $\frac{a\pi\sqrt{b}}{c}$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a + b + c$.

Solution. Rotating $ABCD$ along side BC will form a cone with a smaller cone removed from its base, as shown below.



Note that triangle XDC is an isosceles right triangle, so $XA = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $XD = \frac{4}{\sqrt{2}} = 2\sqrt{2}$. Then the volume of the larger cone will be $\frac{1}{3}\pi(2\sqrt{2})^2 \cdot 2\sqrt{2} = \frac{1}{3}\pi 16\sqrt{2}$, and the volume of the smaller cone will be $\frac{1}{3}\pi(\sqrt{2})^2 \cdot \sqrt{2} = \frac{1}{3}\pi 2\sqrt{2}$. Thus the volume of the solid will be $\frac{1}{3}\pi 16\sqrt{2} - \frac{1}{3}\pi 2\sqrt{2} = \frac{14\pi\sqrt{2}}{3}$. So the answer is $14 + 2 + 3 = \boxed{19}$. △

6. Let $A_1B_1C_1$ be a triangle with interior angles measuring $\angle A_1 = 40°$, $\angle B = 60°$, and $\angle C = 80°$, and let circle O be the circumcircle of $\Delta A_1B_1C_1$. Let A_2 be the point where the angle bisector of interior angle $\angle A_1$ intersects circle O, B_2 be the point where the angle bisector of interior angle $\angle B_1$ intersects circle O, and C_2 be the point where the angle bisector of interior angle $\angle C_1$ intersects circle O. Now, consider triangle $A_2B_2C_2$ and the three points formed by intersecting its three interior angle bisectors with circle O. Call these three points A_3, B_3, and C_3. Similarly, points A_4, B_4, and C_4 are formed by the angle bisectors of $\Delta A_3B_3C_3$ intersecting the circle. This process is repeated until we obtain the points A_{10}, B_{10}, and C_{10}. Find the the measure, in degrees, of the middle (not the largest nor the smallest) of the three interior angles of $\Delta A_{10}B_{10}C_{10}$.

Solution. We first present you a statement that will be proved: If triangle $A_nB_nC_n$ has angles $60 - x$, 60, and $60 + x$, then the angles in triangle $A_{n+1}B_{n+1}C_{n+1}$ is $60 - 0.5x$, 60 and $60 + 0.5x$.

To prove this statement, lets first set angle A_n to $60 - x$, angee B_n to be 60, and angle C_n to be $60 + x$. This means that arc A_nB_n has angle $120 + 2x$, arc A_nC_n has angle 120, and arc B_nC_n has angle $120 - 2x$. We can now find the angles of the next triangle. Since the 3 points in the new triangle bisect the 3 arcs, we can compute arcs $A_{n+1}B_{n+1}$, $A_{n+1}C_{n+1}$, and $B_{n+1}C_{n+1}$. To do this, lets compute the 6 arcs that are formed by the 6 points on the circle. $A_nC_{n+1} = C_{n+1}B_n = 60 + x$, $A_nB_{n+1} = B_{n+1}C_n = 60$, and $C_nA_{n+1} = A_{n+1}B_n = 60 - x$ because the new triangle bisect the arcs formed by the original triangle. Then, we can add certain arcs to get arc $A_{n+1}B_{n+1} = 120 - x$, $A_{n+1}C_{n+1} = 120$, and $B_{n+1}C_{n+1} = 120 - x$. We can convert this back to the angle on the triangle, and when we do that we find that the new triangle has angles $60 - 0.5x$, 60 and $60 + 0.5x$.

Notice that with this statement, no matter what the value of x is, the middle angle always be 60 on the next triangle. As x is halved with each new triangle, the statement can still apply to the next triangle. Since the first triangle can fit the statement with x being 20, we are sure that the middle angle will stay at $\boxed{60}$ no matter how many triangles we draw, so that is our answer.

△

7. An Energizer bunny is placed on the coordinate plane at $(0, 0)$. A move with length x is defined as the bunny moving strictly north, east, south, or west x units. A bunny makes four moves with length 1, 2, 4, and 8 in that order. Find the number of possible coordinate locations the bunny can end up after these four moves.

Solution. Note that the bunny's final x-coordinate will be sum of its east and west moves, and its final y-coordinates will be the sum of its north and south moves. Thus, we can let the bunny make the four moves in any order, and all the possible coordinate locations will be the same.

Now, consider of the binary representations of the four lengths: 1, 10, 100, and 1000. Since each length is a power of two, we can take any subset of the lengths and adding or subtracting its elements will produce unique values.

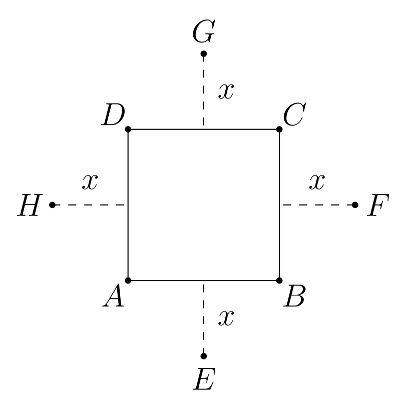
We can then pick which of the moves are north/south (which we'll denote with y) and east/west (x). There are several cases: all east/west moves, $xxxx$; three, $xxxy$; two, $xxyy$; one, $xyyy$; and none, $yyyy$. There are

- $\binom{4}{0} \cdot 2^4 \cdot 2^0 = 16$ ways for $xxxx$,
- $\binom{4}{1} \cdot 2^3 \cdot 2^1 = 64$ for $xxxy$,
- $\binom{4}{2} \cdot 2^2 \cdot 2^2 = 96$ for $xxyy$,

and the remaining cases are symmetric. Thus, there are a total of $16 + 64 + 96 + 64 + 16 = \boxed{256}$ possible coordinate locations for the bunny to end up.

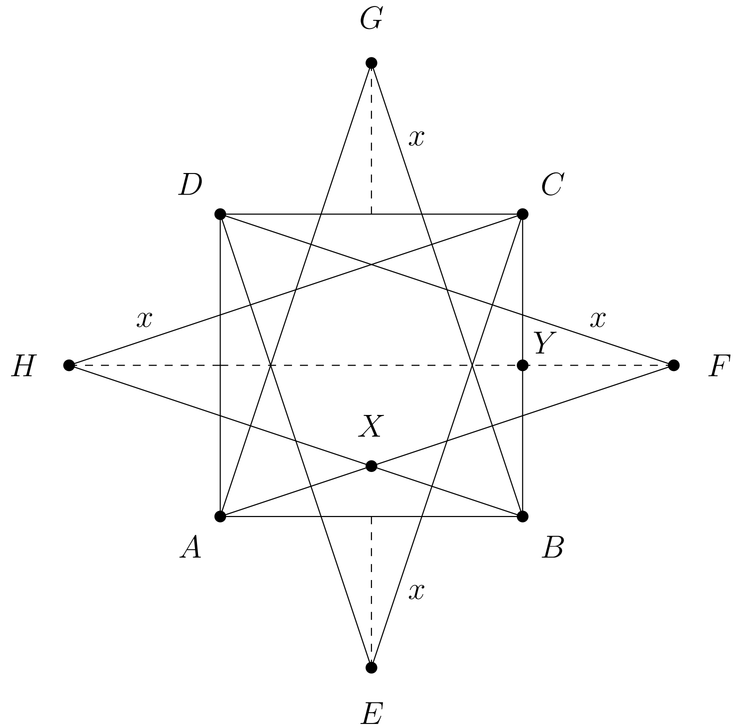
Alternatively, if we recognize that the bunny will never intersect with another path, we can see that there are four possibilities for each move. Consequently, there will be $4^4 = \boxed{256}$ possible final locations. △

8. Let $ABCD$ be a square with side length 2. Construct points E, F, G, and H (outside the square) on the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively such that the distance from each of these points to the respective sides is x units, as shown in the figure below. Let S be the sum of all finite, positive values of x so that the region of overlap between the triangles ΔABG, ΔBCH, ΔCDE, and ΔDAF is a regular polygon. S can be expressed as $\sqrt{a} - b$, where a and b are positive integers and a is not divisible by the square of any prime. Find $a - b$.



Solution. The first key observation is that the desired region of overlap will always be an octagon, no matter the value of x. Secondly, any of these octagons formed will be regular if and only if one of its interior angles measures 135°. This holds because of the convenient symmetry of a square. So armed with these observations, we can tackle the problem in a simple and efficient manner.

We add a few points (namely H and M) and draw $\overline{HM}$ to make the solution cleaner. The figure we'll use is shown below.



In order for $\angle HXF = 135°$, we must have that $\angle ABX = \frac{45}{2}°$, as ΔAXB is isosceles by symmetry. This means that $\angle HBY = \frac{135}{2}$. But by trigonometry, we have that $\angle HBY = \tan^{-1}(2+x)$, as ΔHBY is a right triangle. Hence, we have the equation $\tan\frac{135}{2} = 2 + x$. Using the Half-Angle formula, we get that $\tan\frac{135}{2} = \frac{\sin 135}{1+\cos 135}$. One can compute $\sin 135 = \frac{\sqrt{2}}{2}$ and $\cos 135 = -\frac{\sqrt{2}}{2}$, giving us

$$\begin{aligned}\tan\frac{135}{2} &= \frac{\sin 135}{1+\cos 135}\\ &= \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\\ &= \frac{\sqrt{2}}{2-\sqrt{2}}\cdot\frac{2+\sqrt{2}}{2+\sqrt{2}}\\ &= \frac{2\sqrt{2}+2}{2}\\ &= \sqrt{2}+1.\end{aligned}$$

Thus, we have the equation $\sqrt{2}+1 = 2+x$, yielding $x = \sqrt{2}-1$. So the answer is $2-1 = \boxed{1}$. △

9. For all real y, the expression $x^3 + 2x^2 + 2yx^2 - 4x + xy^2 + 8 - y^2$ is greater than or equal to 0 for all $x \geq c$. Find the minimum possible value of c.

Solution. Rewriting the expression as a quadratic in y results in the expression $(x-1)y^2 + (2x^2)y + (x^3 + 2x^2 - 4x + 8)$. Recall that for a quadratic $P(x)$ to be greater than or equal to 0 for all real values x, its discriminant must be less than or equal to 0. In addition, we must have a positive leading coefficient (and we may only have a leading coefficient of 0 when $P(x)$ is a constant function).
First, let us calculate the determinant of our quadratic in y. For the quadratic $ax^2 + bx + c$, the value of the discriminant is $b^2 - 4ac$. Using this formula, the discriminant for our quadratic in y is

$$\begin{aligned}(2x^2)^2 - 4(x-1)(x^3 + 2x^2 - 4x + 8) &= 4x^4 - 4(x^4 + x^3 - 6x^2 + 12x - 8)\\ &= -x^3 + 6x^2 - 12x + 8\\ &= -(x-2)^3.\end{aligned}$$

Therefore, we must have $-(x-2)^3 \leq 0$, or $(x-2)^3 \geq 0$ which is equivalent to $x \geq 2$.
Finally, we must have the leading coefficient greater than 0, giving us $x > 1$. If $x = 1$, then our quadratic becomes $2y + 8$ which is not greater than 0 for all real y. Therefore, the original expression is greater than or equal to 0 for all y when $x > 1$ and $x \geq 2$, which is equivalent to $x \geq 2$. Hence, the minimum possible value of c is $\boxed{2}$. △

10. Given $ab + bc + ca = 2$, find the sum of all possible values of $a + b + c$ if a, b, and c are real numbers satisfying

$$a^3 - a^2 + b^3 - b^2 + c^3 - c^2 = ab(3c + 2) + 2c(a + b).$$

Solution. First we expand the RHS of the equation:

$$a^3 - a^2 + b^3 - b^2 + c^3 - c^2 = 3abc + 2ab + 2bc + 2ac.$$

Next, we rearrange the terms:

$$a^3 + b^3 + c^3 - 3abc = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac.$$

Both the LHS and RHS can be factored:

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) = (a + b + c)^2$$

Since we are given $ab + bc + ca = 2$, we can rewrite the LHS:

$$(a + b + c)((a + b + c)^2 - 3 \cdot 2) = (a + b + c)^2$$

Now, let $a + b + c = x$. Our equation then becomes

$$x(x^2 - 6) = x^2 \implies x^3 - x^2 - 6x = 0 \implies x(x-3)(x+2) = 0,$$

which has roots $x = -2, 0, 3$. So the possible values of $a+b+c$ are -2, 0, and 3. Now let's see if they actually work. If $a+b+c = 0$, we can use the fact that $ab+bc+ca = 2$ to write

$$\begin{aligned} 0 &= (a+b+c)^2 \\ &= a^2+b^2+c^2+2(ab+bc+ca) \\ &= a^2+b^2+c^2+4 \\ &\geq 4, \end{aligned}$$

as squares are non-negative. This is a contradiction; hence $a+b+c$ cannot equal 0. Likewise, if $a+b+c = -2$, then

$$\begin{aligned} 4 &= (a+b+c)^2 \\ &= a^2+b^2+c^2+2(ab+bc+ca) \\ &= a^2+b^2+c^2+4 \\ &\geq 4. \end{aligned}$$

This implies that $a = b = c = 0$, because the sum of the squares of variables is zero if and only if all the variables are zero. However, $a = b = c = 0$ doesn't satisfy $ab+bc+ca = 2$. Hence, $a+b+c$ cannot equal -2. It turns out we can have $a+b+c = 3$, as one solution is just $(a,b,c) = (2,1,0)$ (we can verify this by noting $ab+bc+ca = 2$). Therefore, our answer is $\boxed{3}$. △