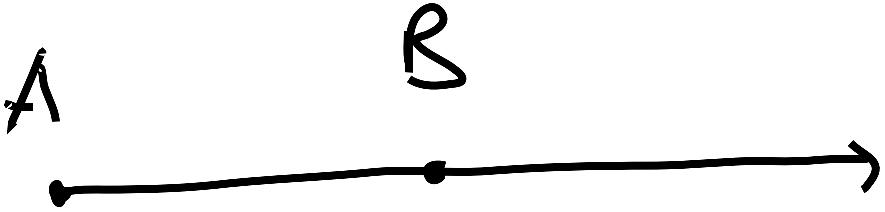
Week 9:
Angles and Triangles

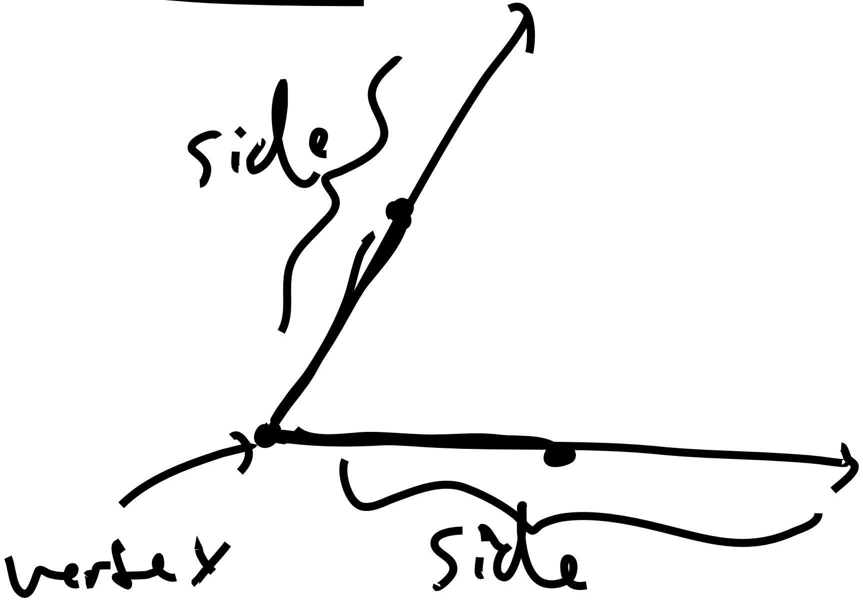
Angles

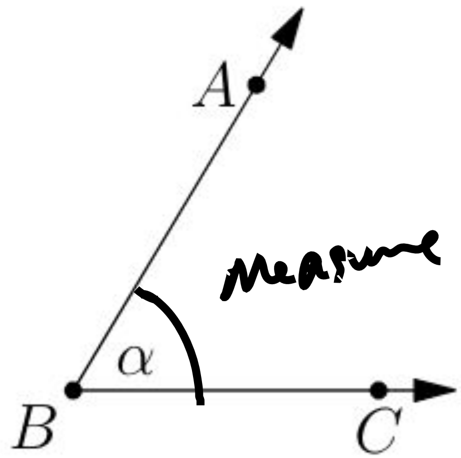
Definition. A *ray* is a line segment extended infinitely in one direction.



$\overrightarrow{AB}$

Definition. An *angle* as the union of two rays that share an endpoint. We all this common endpoint the *vertex* of the angle and refer to each of the two rays as a *side* of the angle. We can also use line segments as part of a side of an angle.





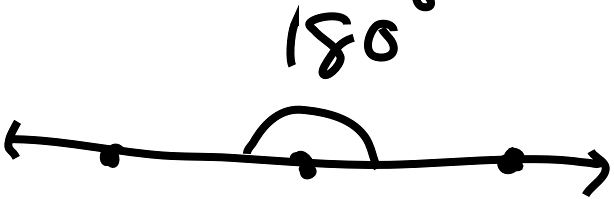
measure

$\angle ABC = \alpha^{\circ}$

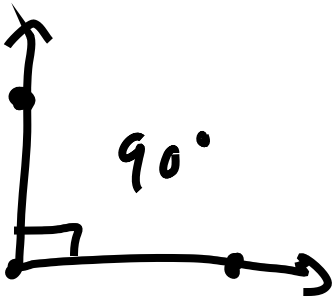
α - degrees

- radians

Straight angle:

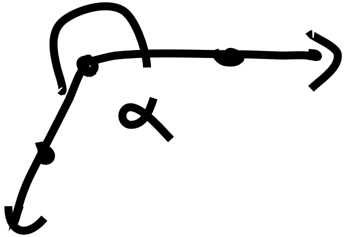


Right angle:



Reflex angle:

$>180^\circ$



$360^\circ - \alpha$

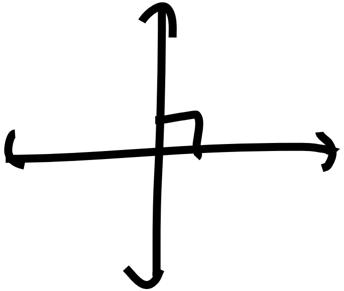
Definition. Two angles with the same measure are called congruent angles.

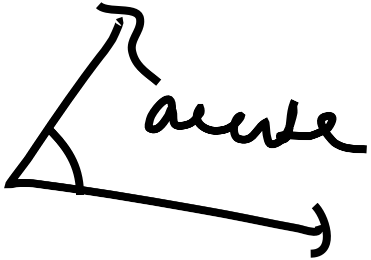
Definition. Two angles are *complementary* if their measures sum to 90°. Additionally, two angles are *supplementary* if their sum is 180°.

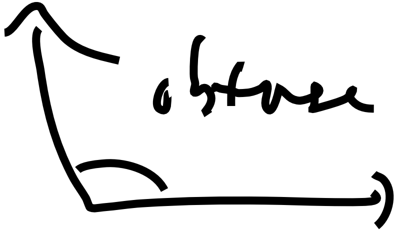
Definition. An *angle bisector* is a line through an angle that splits it into two congruent angles.

Definition. Any two intersecting lines that form an angle measuring 90° (i.e. a right angle) are called *perpendicular lines*. The symbol ⊥ is used to denote perpendicularity, similar to the parallel symbol ∥ being used to indicate parallel lines.

Definition. An angle with measure strictly between 0° and 90° is called *acute*, whereas an angle with measure strictly between 90° and 180° is called *obtuse*.





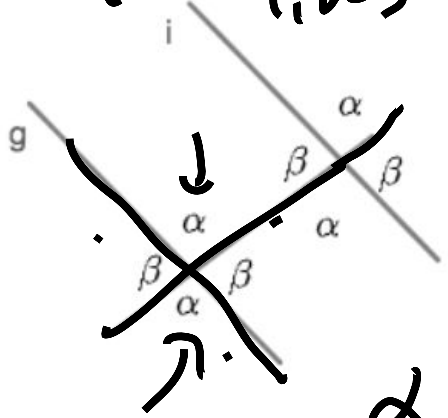


If a line intersects a pair of parallel lines (i.e. two lines that never intersect), then there are several pairs of congruent angles:

$g \parallel i$

Transversal:

line intersects two parallel lines



Vertical angles:

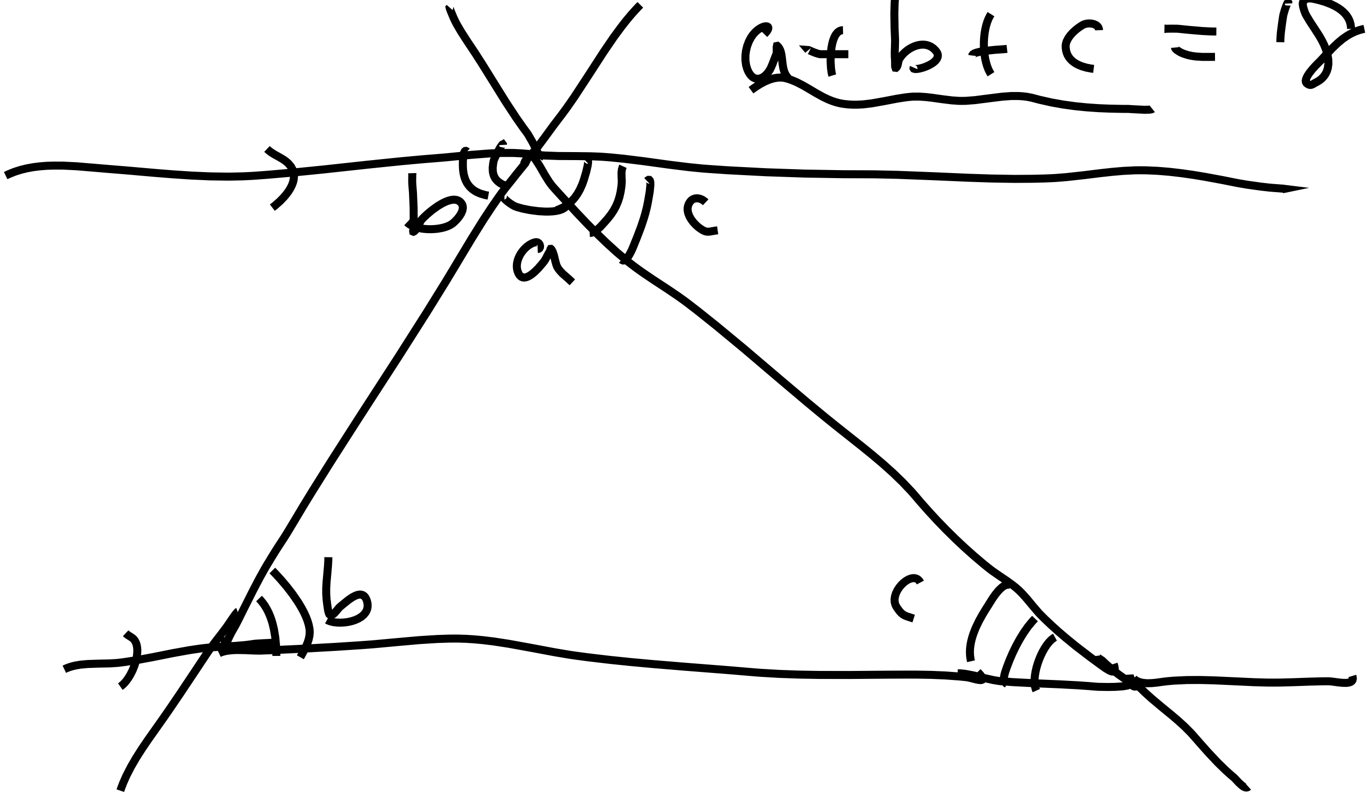
$\alpha = \alpha$

Theorem 12.1. *Two lines are parallel if and only if any transversal (or segment) intersecting both lines is incident at the same angle.*

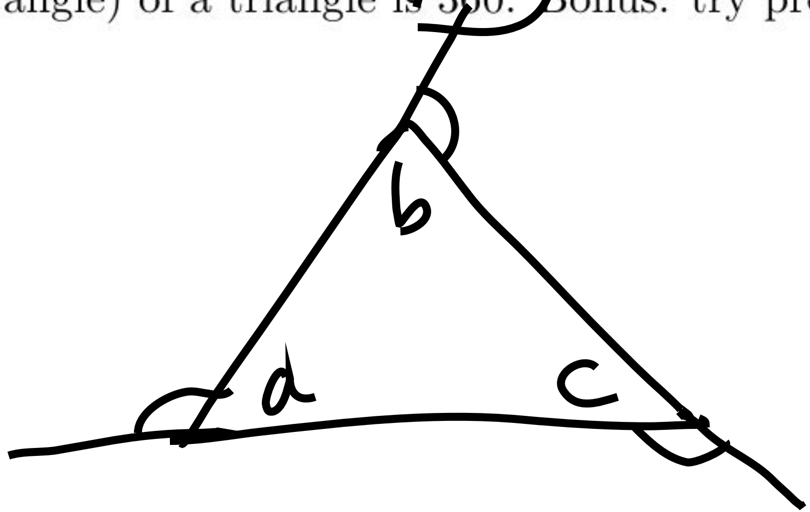
Triangles and Basic Properties

Theorem 12.2. *The sum of the interior angles in a triangle is* 180°.

$a + b + c = 180°$



Checkpoint 12.1. Prove that the sum of the exterior angles(the supplementary angle of an interior angle) of a triangle is 360. Bonus: try proving this for any n-sided polygon!



$$a+b+c=180$$

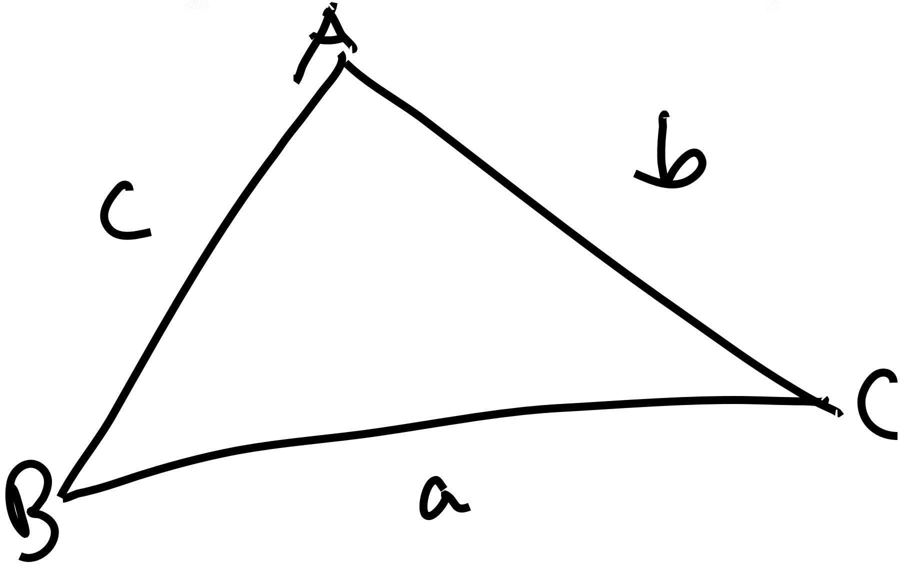
$$180-a+180-b+180-c$$

$$=3\cdot 180-(a+b+c)$$

$$=3\cdot 180-180=360$$

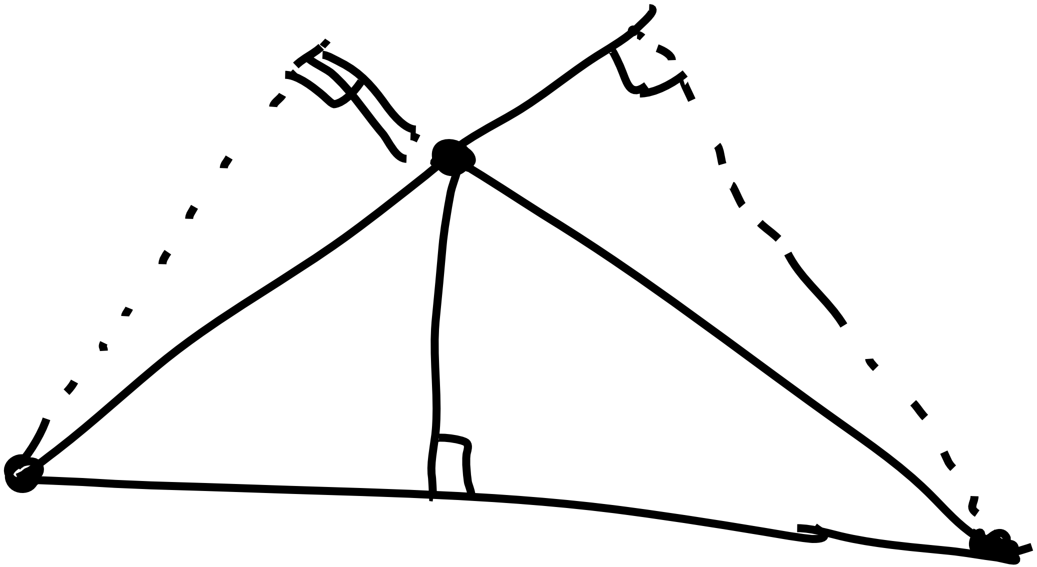
Theorem 12.3. *If $\angle A > \angle B > \angle C$, then $a > b > c$, where $a = BC, b = CA$, and $c = AB$.*

This can be proved with the Law of Sines, where $\frac{a}{\sin A} = \frac{b}{\sin B}$.

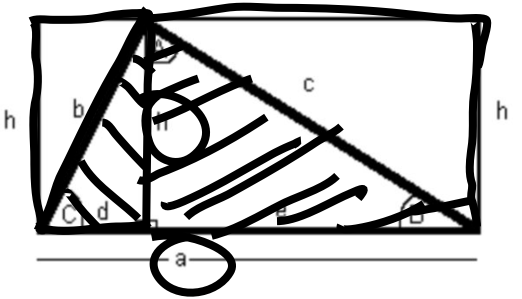


$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Definition. The altitude of a triangle is a line segment that extends from one vertex of the triangle and hits the opposite side at a 90° angle. Every triangle has three altitudes - one from each vertex.



Theorem 12.4. *The area of any triangle is $\frac{1}{2}bh$, where h is the length of any altitude of the triangle and b is the length of the side that this altitude hits (i.e. the side opposite to the vertex associated with the altitude). h stands for height and b stands for base.*



$\frac{1}{2}ah$

Checkpoint 12.2. A triangle has side lengths 4, 6, and 7. Find the ratio between the longest altitude and the shortest altitude of the triangle.

$A = \frac{1}{2}bh$

$\frac{2A}{b} = h \leftarrow$ [illegible]

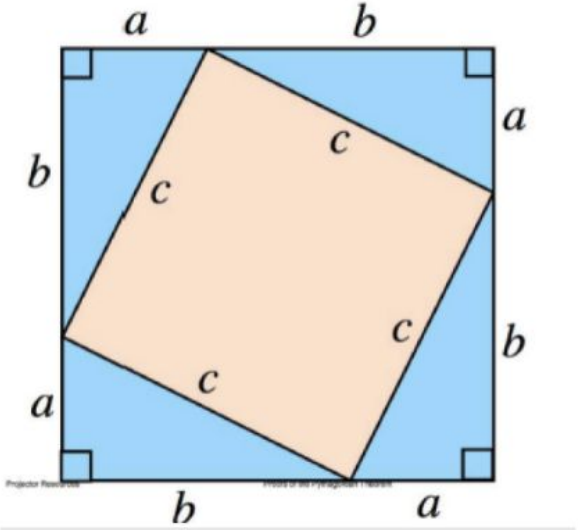
$\frac{2A}{4}, \frac{2A}{6}, \frac{2A}{7}$

$\frac{\frac{2A}{4}}{\frac{2A}{7}} = \frac{7}{4}$

Right Triangles

Definition. A right triangle is a triangle that contains a right angle. The hypotenuse of a right triangle is the longest side of the triangle, namely, the side opposite the right angle. The two legs of a right triangle are the non-hypotenuse side; they are the two sides that are incident with the right angle.

Theorem 12.5. *For any right triangle, let c be its hypotenuse and a and b be its two legs. Then we have $a^2 + b^2 = c^2$, famously known as the Pythagorean Theorem.*



$$(a+b)^2 = a^2 + 2ab + b^2$$

$$4 \cdot \cancel{\frac{1}{2}ab} + c^2 = a^2 + \cancel{2ab} + b^2$$

$$a^2 + b^2 = c^2$$

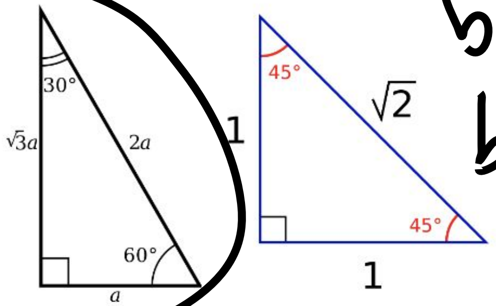
Definition. A $30-60-90$ triangle refers to any triangle with angles $30, 60,$and 90. The ratio between the hypotenuse and the shorter leg is 2 and the ratio between the longer leg and the shorter leg is $\sqrt{3}$, as shown in the figure below.

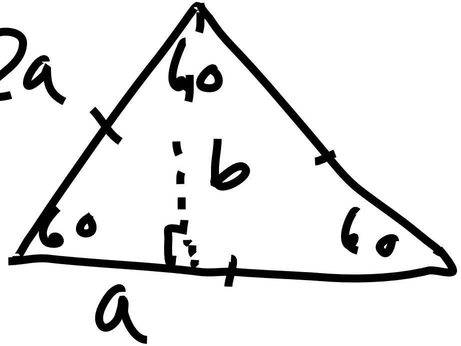
Definition. A $45-45-90$ triangle refers to any triangle with angles $45, 45,$and 90. Note that any such triangle is isosceles, as two of its angles are the same; in particular, the two legs of the right triangle are congruent. The ratio between the hypotenuse and any leg of the triangle is $\sqrt{2}$.

$a^2+b^2=(2a)^2$

$b^2=3a^2$

$b=\sqrt{3}a$





Checkpoint 12.3. Verify the ratios stated above for a $30-60-90$ triangle, by dropping down an altitude in an equilateral triangle and noting that this altitude splits the triangle into two $30-60-90$ triangles.

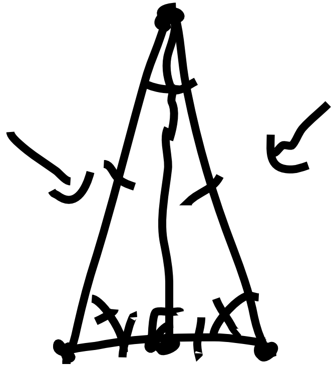
Equilateral and Isosceles Triangles

Definition. A triangle is isosceles if it has two sides with the same length or two congruent angles. If a triangle is isosceles, both of these conditions will be satisfied. In particular, the angles opposite to the equal sides will be congruent. These angles are called the base angles.

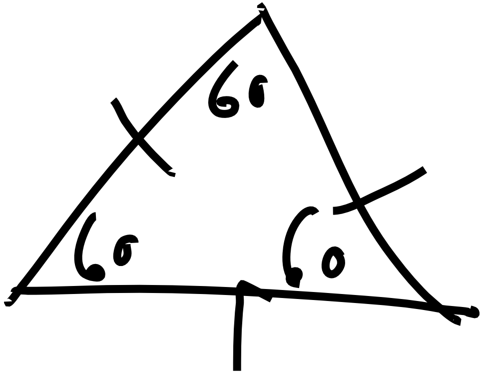
Theorem 12.6. *The altitude from the non-base angle divides isosceles triangle into two congruent triangles.*

Corollary 12.6.1. *The altitude of an isosceles triangle from the non-base angle is also the median and angle bisector from that vertex. In other words, the al*

If we know which two sides of an isosceles triangle have the same length, then we can identify which two angles are congruent.

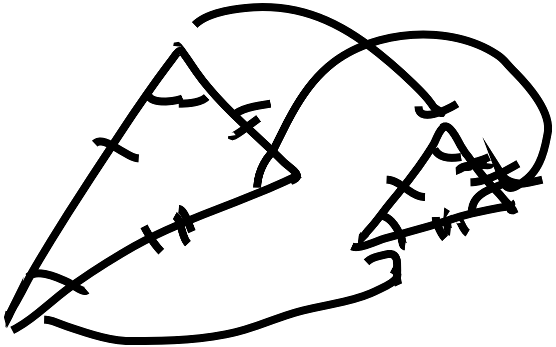


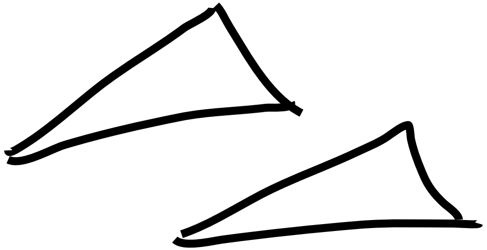
Definition. A triangle is equilateral if all three of its sides are the same length, or if all three of its angles are $60°$. If either of these conditions are fulfilled, the triangle is equilateral.



Similarity and Congruence

Similarity and congruence are also two very powerful tools for angle chasing. Two triangles are similar when the dilation of one of the triangles is congruent to the other. In other words, two triangles are similar when they have the same angles and their corresponding sides are in proportion. However, we don't need to check that all sides are in proportion and that the angle measures are equal. There are special rules we can use to quickly tell whether triangles are congruent.





SSS Similarity

S = side

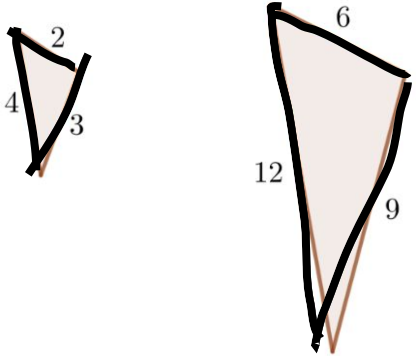
Two triangles are similar if their corresponding sides are in proportion. This means that if

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

then ΔABC is similar to ΔDEF.

The following two triangles are similar from SSS Similarity:

$$\frac{4}{12} = \frac{2}{6} = \frac{3}{9}$$

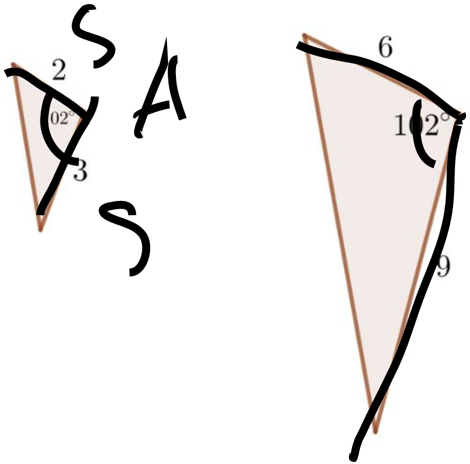


SAS Similarity

A = angle

Two triangles are similar if two sets of sides are in proportion, and the angle between them are equal in the two triangles.

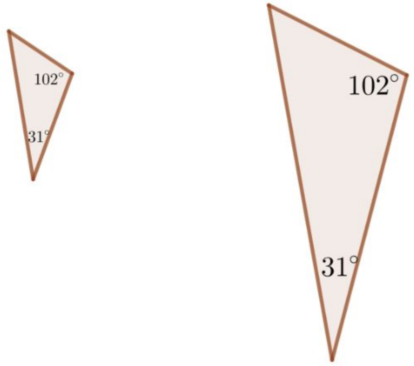
The following two triangles are similar from SAS Similarity:



AAA (or AA) Similarity

Two triangles are similar if they have the same angles. However, if we know two angles in a triangle, we know the third: x, y, and $180 - x - y$. Hence, we only need that the two triangles have two angles in common (AA).

The following two triangles are similar from AA Similarity:

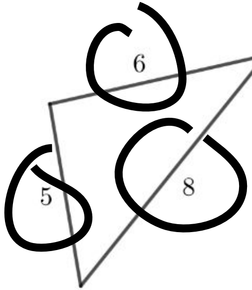


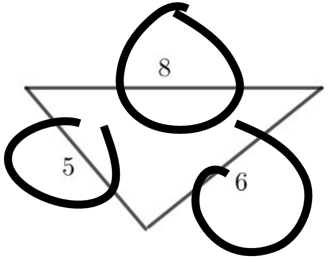
In addition, two triangles are congruent if they are the "same" - more technically, they have the same angle measures and same side lengths. This means that if two triangles are similar and have side lengths in a ratio of 1:1, then there are congruent. As a result, we can use a few special variations of the similarity rules to determine congruence.

Congruency

SSS Congruence says that if two triangles have the same side lengths, then they are congruent. In other words, having the same side lengths implies that the two also have the same angle measures, hence, they are congruent.

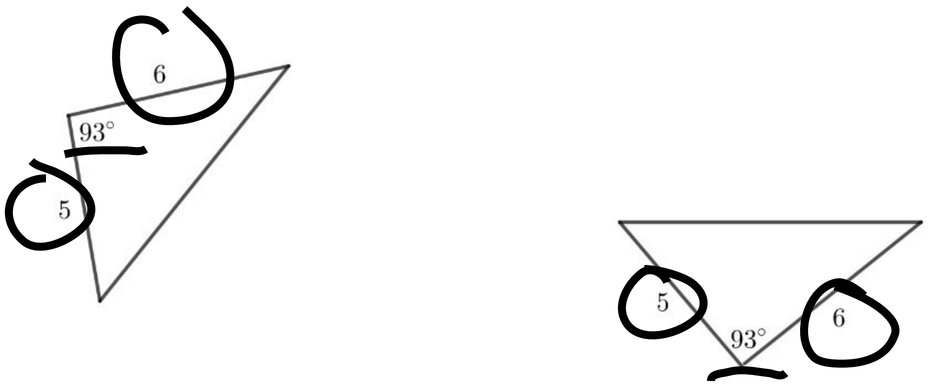
The following two triangles are congruent by SSS Congruence:





SAS Congruence says that if two triangles have two sides that are the same length and the angle between the two sides are equal, then the two triangles are congruent.

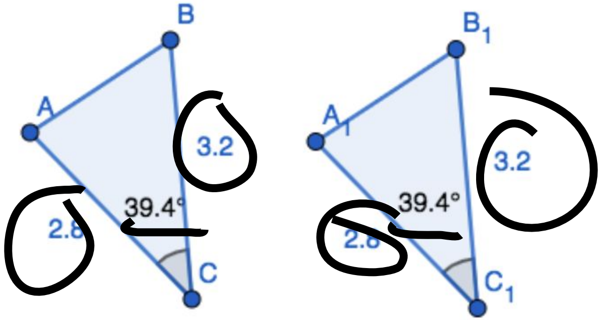
The following triangles are congruent by SAS Congruence:



ASA Congruence

If two triangles have two angles that are congruent and if the sides in between the two angles have the same length, the two triangles are congruent.

The following pair of triangles are congruent by ASA Congruence:



SAA Congruence

SAA Congruence says that if two triangles have two angles and a side length in common, then they are congruent. The common side must be in the same position relative to the two common angles.

The following pair of triangles are congruent by SAA Congruence:

SAA Congruence is equivalent to ASA Congruence because you can find the measure of the other angle adjacent to the side length. We actually recommend ASA Congruence over SAA Congruence because SAA Congruence can get confusing.

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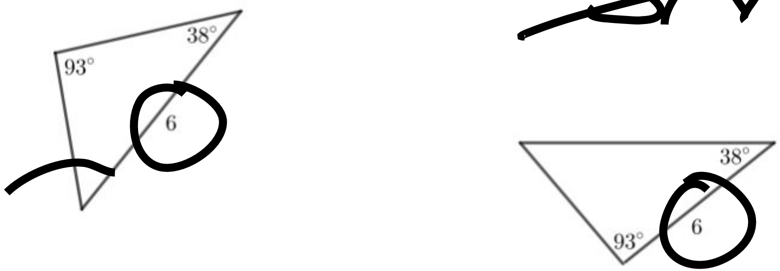
Checkpoint 12.4. The following triangles are NOT congruent - why?

~~SAA~~



ASA

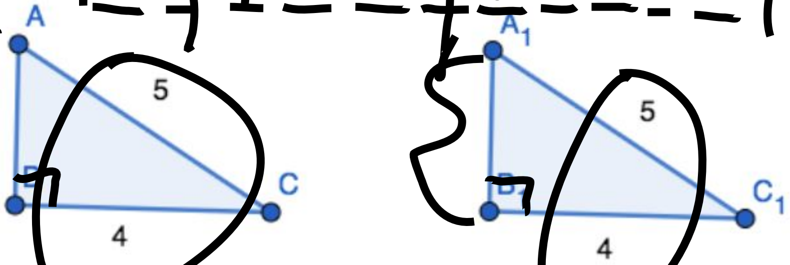
Again, SAA Similarity is the same as AA Similarity because we know that two of the angles are congruent.

12.3.5 HL Congruence

Two right triangles are congruent if they have a congruent hypotenuse and leg.

Pythagorean Theorem

SSS



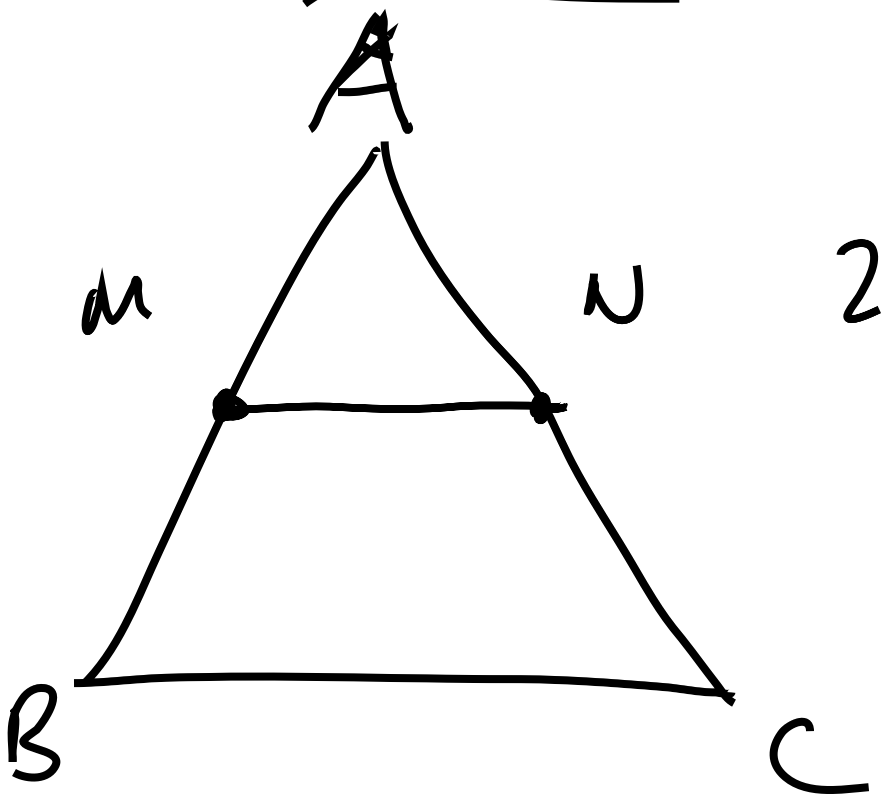
$a^2+b^2=c^2$

For example, if $\angle ABC = \angle A_1B_1C_1 = 90°$, then ΔABC and $\Delta A_1B_1C_1$ are congruent by HL congruence because their hypotenuses are both 5, and one of their legs are 4.

If two right triangles' hypotenuse and leg are in proportion, then they will be similar. However, HL Similarity isn't as useful as HL Congruency.

Checkpoint 12.5. Why is HL congruence true?

Example 12.1. In triangle ABC, M is the midpoint of AB and N is the midpoint of AC. Prove that the length of MN is $\frac{1}{2}BC$.



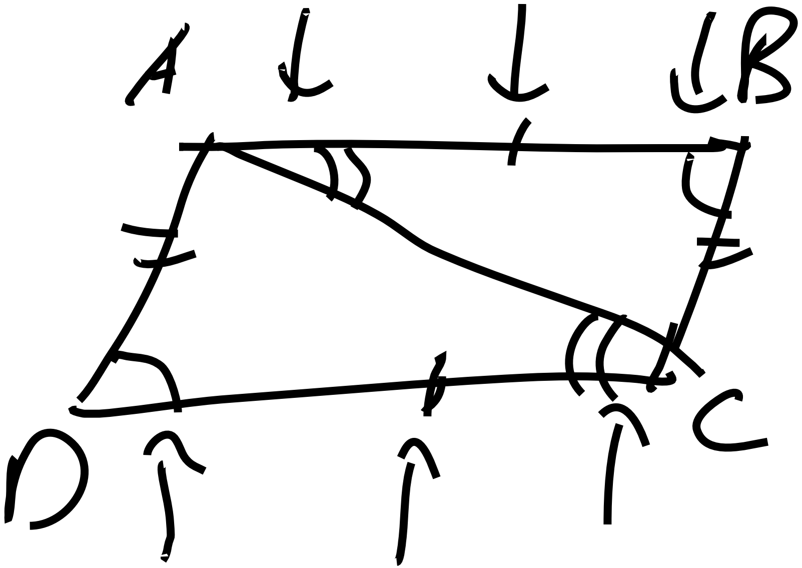
$$\triangle AMN \sim \triangle ABC$$

$$2AM = AB$$

$$\frac{AM}{AB} = \frac{1}{2}$$

$$\frac{MN}{BC} = \frac{1}{2} \Rightarrow MN = \frac{1}{2}BC$$

Example 12.2. Let $ABCD$ be a parallelogram. Show that ΔABC is congruent to ΔCDA using SSS, SAS, and SAA Congruence.



$AB = DC$

$AD = BC$

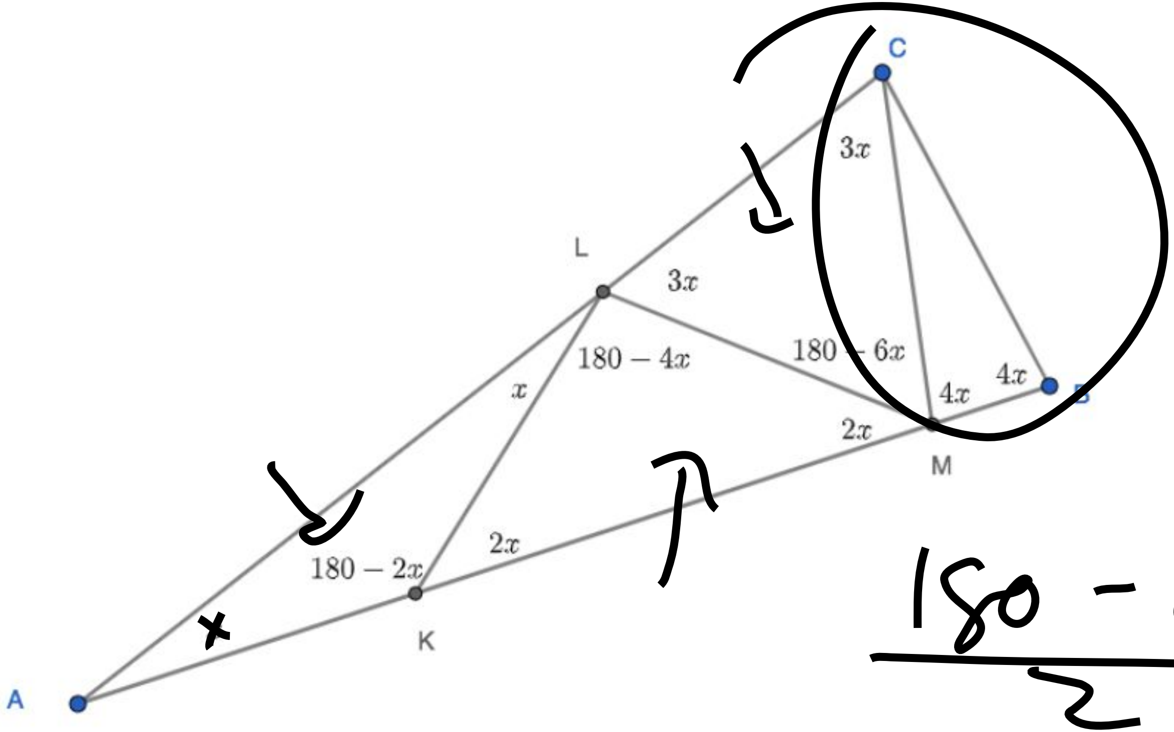
AC

12.4 Angle Chasing

Angle chasing is the art of using geometric tools to find the measures of angles in a figure. The most helpful thing you can do to solve angle chasing problems (or any geometry problem) is to draw a large, neat diagram! We highly recommend getting a straightedge or a ruler to help keep your lines straight. Drawing tidy diagram will help you spot geometric figures or relationships easier, which could quickly unravel a problem or give you a hunch that guides how you will proceed. Some common examples of these include congruent segments and angles, similar and congruent triangles, right angles, isosceles and equilateral triangles, and parallel lines.

Another technique that aids with angle chasing is *geometric construction*. In this technique, we draw new auxiliary lines or extend some of the lines in the diagram in order to expose more of the geometry. Some common auxiliary lines to draw are parallel lines, angle bisectors, and altitudes.

Example 12.3. Let ABC be a triangle with $AB = AC$ and let K and M be points on the side AB and L a point on the side AC such that $BC = CM = ML = LK = KA$. Find $\angle A$.
Source: 106 Geometry



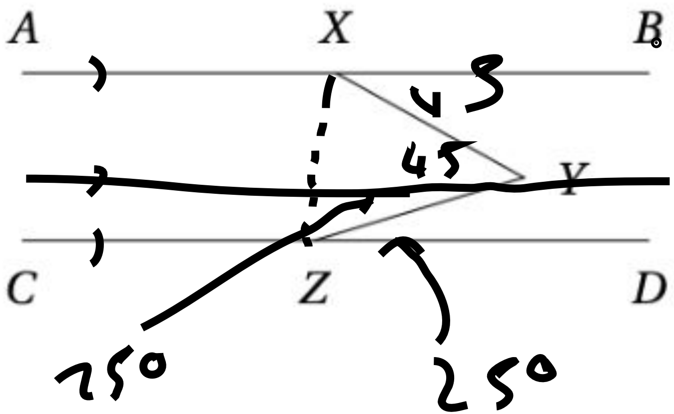
$180 - x = 8x$

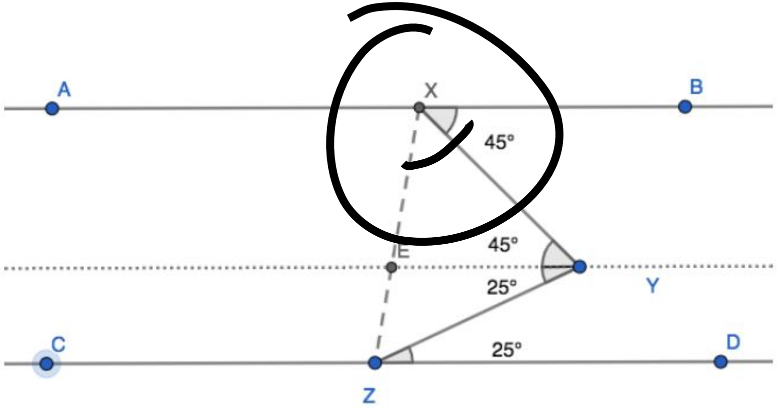
$9x = 180$

$x = 20$

$\frac{180 - x}{2} = 4x$

Example 12.4. In the figure below, $\overline{AB}$ is parallel to $\overline{CD}$, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. What is the degree measure of $\angle YXZ$? *Source: Purple Comet*





$25 + 45 = 70°$

$$\frac{180 - 70}{2} = 55°$$