Week 7:
Recursion

1. Recursive Sequences

A *recursive* sequence or function, also known as a *recurrence relation*, is defined using previous values of the function. For example, the Fibonacci sequence is a recursive sequence and can be defined as $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. We see that

$$
\begin{aligned}
F_2 &= 1 + 0 = 1 \\
F_3 &= 1 + 1 = 2 \\
F_4 &= 2 + 1 = 3 \\
F_5 &= 3 + 2 = 5, \\
F_6 &= 8 \\
F_7 &= 13 \\
F_8 &= 21
\end{aligned}
$$

Checkpoint 1.1. Find the 5th term in the following recursive sequence: $a_1 = 3, a_n = 2a_{n-1} + 3$

n	a_n
1	3
2	9
3	21
4	45
5	93

In a later section, we will discuss how to find a closed-form expression (i.e. one that doesn't involve other terms of the sequence) for the n-th term of a homogeneous recursive sequence (this will also be defined later). For other types of recurrence relations, we may be able to find a closed-form for the n-th term. For example, consider the recurrence defined by $a_0 = 1$ and $a_n = n \cdot a_{n-1}$ for $n \in \mathbb{Z}^+$. We see that

$$
\begin{aligned}
a_n &= n \cdot a_{n-1} \\
&= n(n-1)a_{n-2} \\
&= n(n-1)(n-2)a_{n-3} \\
&\ \ \vdots \\
&= n(n-1)(n-2)\cdots 1 \\
&= n!
\end{aligned}
$$

Checkpoint 1.2. Find a closed-form expression for the n-th term of the recursive sequence defined by $a_1 = 2$ and $a_n = n(n-1) \cdot a_{n-1}$

$$a_n = n(n-1)(n-1)(n-2)a_{n-2}$$

$$= n(n-1)(n-1)(n-2)(n-2)(n-3)a_{n-3}$$

$$\vdots$$

$$= n(n-1)^2(n-2)^2 \cdots 2^2$$

$$= n((n-1)!)^2$$

Example 1.1. Find a closed form for the sequence $a_1 = 4$, $a_n = a_{n-1} - 5$.

n	a_n
1	4
2	-1
3	-6
4	-11
5	-16

$$d = -5$$

$$a_n = 4 - 5(n-1)$$

Checkpoint 1.3. Find a closed form for the sequence $a_1 = 7$, $a_n = 5 * a_{n-1}$.

n	a_n
1	7
2	$7 \cdot 5$
3	$7 \cdot 5^2$
4	$7 \cdot 5^3$

$$a_n = 7 \cdot 5^{n-1}$$

2. Solving Cyclic Recursions

We can find a explicit representation for certain recursive functions, which is often called a closed form. An explicit representation means that we can directly plug in n to find the value of $f(n)$ without needing to know any of the previous values of f. This can be a complicated process, so lets take a look at some basic examples first.

Sometimes a problem involves a *periodic* recurrence. This means that the values produced by the recursive relation will repeat regularly. In cases like this, solving the recursive equation is usually not necessary. Instead, we can find the pattern and use it to obtain the answer.

Example 1.2. Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$$

for all $n \geq 3$. Find t_{2020}. *Source: AIME*

$t_1 = a \quad t_2 = b$

$t_3 = \dfrac{5b+1}{25a}$

$t_4 = \dfrac{\frac{5b+1}{5a} + 1}{25b} = \dfrac{5a+5b+1}{125ab}$

$t_5 = \dfrac{\frac{5a+5b+1+25ab}{25ab}}{\frac{5b+1}{a}} = \dfrac{5a+1}{25b}$

$t_6 = a \quad t_7 = b$

$\dfrac{5 \cdot 20 + 1}{25 \cdot 21} = \dfrac{101}{525}$

Checkpoint 1.4. Let a_n be the remainder when F_n is divided by 3 (equivalently, $a_n = F_n \mod 3$), where F_n is the Fibonacci sequence, defined by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$. Determine the value of a_{2020}.

0, 1, 2

n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	1	2	0	2	2	1	0	1

$2020 = 8 \cdot k + 4$

$a_{2020} = \boxed{0}$

3. Solving Linear Homogeneous Recursions

A *linear homogeneous recurrence relation* is a recurrence relation defined by

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m},$$

for all $n \geq k$, where the c_i's are real constants and m and k are non-negative integers with $k \geq m$ and $c_m \neq 0$.

In other words, a linear homogeneous recurrence is one that is linear, homogeneous, and has constant coefficients. This sequence is linear since all the a_i's on the right-hand side are raised to the power one (not squared, cubed, etc.). This relation is also homogeneous since every term on the right-hand side all have the same degree (namely, 1). If you are not familiar with the phrase "degree of a term", it simply refers to the sum of the exponents of the variables in the term. To see if you have a grasp of this definition, try the following example.

Example 1.3. Which of the following recurrence relations are linear homogeneous recurrence relations? For the ones that are not, which of the three criteria (linearity, homogeneity, and constant coefficients) do they not meet?

L

H C

1. $a_n = na_{n-1}$ L, H
2. $a_n = a_{n-1}a_{n-2}$ H, C
3. $a_n = ca_{n-1}$ for some constant c. L, H, C
4. $a_n = 2a_{n-1} + a_{n-2} + 3$ L, C
5. $a_n = a_{n-1}^2 + a_{n-2}^2$ H, C
6. $a_n = 2a_{n+1} - 3a_{n-1}$ $a_{n+1} = 0.5a_n + 1.5a_{n-1}$ L, H, C
7. $a_n = 3a_{n-1}^2 + 2a_{n-2}$ C
8. $a_n = 2a_{n-1} - 3a_{n-2}$ L, H, C

Now, we'll introduce another definition. The *characteristic polynomial* of the linear homogeneous recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m}$$

$a_n = a_{n-1} + a_{n-2}$

is the polynomial

$$p(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1} x - c_m.$$

Theorem 1.1. *Suppose we have the recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m}$$

with characteristic polynomial

$$p(x) = x^m - c_1 x^{m-1} - c_2 x^{m-2} - \cdots - c_{m-1} x - c_m.$$

Then if $p(x)$ has distinct roots $r_1, r_2, \ldots, r_m$, any sequence $\{a_n\}$ satisfies this recurrence if and only if

$$a_k = b_1 r_1^k + b_2 r_2^k + \cdots + b_m r_m^k,$$

for all $k = 0, 1, \ldots, n$, where $b_1, b_2, \ldots, b_m$ are constants.

Example 1.4. Find a closed-form expression for the n-th term of the Fibonacci sequence.

$$F_n = F_{n-1} + F_{n-2} \qquad \boxed{F_0 = 0 \quad F_1 = 1}$$

$$p(x) = x^2 - x - 1 \qquad \frac{1+\sqrt{5}}{2}, \ \frac{1-\sqrt{5}}{2}$$

$$F_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{cases} b_1 + b_2 = 0 \\ b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases} \qquad b_1 = \frac{1}{\sqrt{5}} \quad b_2 = -\frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Checkpoint 1.4. Define the Pell sequence, $\{P_n\}$, as follows:

$$P_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2P_{n-1} + P_{n-2} & \text{if } n > 1. \end{cases}$$

$$\boxed{P_n = \frac{1}{2\sqrt{2}}(1+\sqrt{2})^n - \frac{1}{2\sqrt{2}}(1-\sqrt{2})^n}$$

Find a closed form for P_n.

$p(x) = x^2 - 2x - 1$

$1+\sqrt{2}$ $\quad$ $1-\sqrt{2}$

$P_n = b_1(1+\sqrt{2})^n + b_2(1-\sqrt{2})^n$

$$\begin{cases} b_1 + b_2 = 0 \\ b_1(1+\sqrt{2}) + b_2(1-\sqrt{2}) = 1 \end{cases}$$

$b_1 = \frac{1}{2\sqrt{2}}$

$b_2 = -\frac{1}{2\sqrt{2}}$

Example 1.5. The sequence $\{a_n\}$ is defined recursively by $a_0 = 1$, $a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer? *Source: AMC*

$$\log_2 a_n = \log_2 a_{n-1} + 2\log_2 a_{n-2}$$

$$\boxed{b_n = 19\log_2 a_n} \qquad b_0 = 0, \quad b_1 = 1$$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + 2c_2 = 1 \end{cases}$$

$$b_n = b_{n-1} + 2b_{n-2}$$

$$c_1 = -\frac{1}{3}$$

$$c_2 = \frac{1}{3}$$

$$p(x) = x^2 - x - 2 = (x+1)(x-2)$$

$$b_n = c_1(-1)^n + c_2(2)^n$$

$$b_n = -\frac{1}{3}(-1)^n + \frac{1}{3}2^n$$

$$b_n = \frac{2^n - (-1)^n}{3} \qquad a_n = 2^{b_n/19}$$

$$a_1 \cdot a_2 \cdots a_k = 2^{b_1/19}\, 2^{b_2/19} \cdots 2^{b_k/19}$$

$$= 2^{b_1/19 + b_2/19 + \cdots b_k/19}$$

$b_1 + b_2 + \cdots b_k$: k is even

$$= 2 - (-1) + 2^2 - 1 + 2^3 + 1 \cdots 2^n - 1$$

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

$$= \frac{2 + 2^2 + \cdots 2^n}{3} = \frac{2^{n+1} - 2}{3}$$

$$b_1 + b_2 + \cdots + b_k = 2-(-1) + 2^2 - 1 + 2^3 + 1 \cdots 2^k + 1$$

$$= \frac{1+2+2^2+\cdots 2^k}{3}$$

$$= \frac{2^{k+1}-1}{3}$$

K is odd

$$2^{18} \equiv 1 \pmod{19}$$

$$\Rightarrow k = 17$$

$$2^{18} - 1 \equiv 0 \pmod{19}$$

$$2^{19} - 2 \equiv 0 \pmod{19}$$

$$2 \cdot 2^{18} \equiv 2 \pmod{19} \Rightarrow 2$$

$$\Rightarrow k = 18$$

4. Applications in Combinatorics

Recursion can be a powerful technique to represent combinatorics problems in a simpler fashion and make solving them more straightforward. To do recursion in such problems, we will first have to find the base case that is easy to compute. Then, we can find the recursive step, which the relationship between one(which could be the base) case and the next case.

Example 1.5. Find the number of 10 digit positive binary numbers that do not have a pair of consecutive 0s.

a_n = n digits, end in 0

b_n = n digits, end in 1

1 10 101 11

$$a_n = b_{n-1} \qquad b_n = a_{n-1} + b_{n-1}$$

Example 1.5. Find the number of 10 digit positive binary numbers that do not have a pair of consecutive 0s.

n	a_n	b_n
1	0	1
2	1	1
3	1	2
4	2	3
5	3	5
6	5	8
7	8	13
8	13	21
9	21	34
10	34	55

$$a_n = b_{n-1}$$

$$b_n = a_{n-1} + b_{n-1}$$

$a_{10} = 34$ $\quad$ $b_{10} = 55$

$$\boxed{89}$$

Example 1.6. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14? *Source: AIME*

n	a_n	b_n
0	1	1
1	0	1
2	1	0
3	1	2
4	1	1
5	3	3
6	2	4
7	6	5
8	6	10
9	11	11
10	16	21
11	22	27
12	37	43
13	49	64
14	80	92

Checkpoint 1.5. Jack is jumping up a flight of stairs. He can take one step or two steps in a jump. Find the number of ways he can climb up n steps, if $n = 10$. *Source: Tristan Shin*

a_n

$a_n = a_{n-1} + a_{n-2}$

$a_1 = 1$

$a_2 = 2$

$n-1 + 1 = n$

$n-2 + 2 = n$

a_{10}

1 2 3 5 8 | 13 21 34 55 89

5. Applications in Functional Equations

Example 1.7. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$? *Source: AMC*

Example 1.7. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$? *Source: AMC*