Iowa City Math Circle Handouts
Logarithms and Exponents

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1 Basics of Exponentiation

In this week, we will deal with expressions involving the form $a^b = a \times a \times \cdots \times a$, where there are $b-1$ multiplication signs. We call a the *base* and b the *exponent*. We will first discuss a set of results collectively known as the *Law of Exponents*.

Theorem 1.1. *(Law of Exponents) The following properties govern arithmetic with exponents.*

1. $a^0 = 1$ *for all* $a \neq 0$*. The value of* 0^0 *is commonly chosen to be* 1*, but sometimes it's chosen to be undefined for certain applications.*
2. $0^b = 0$ *for all* $b \neq 0$*.*
3. $a^{-b} = \frac{1}{a^b}$ *for all* $a \neq 0$*.*
4. $a^b \cdot a^c = a^{b+c}$ *for all* $a \neq 0$*.*
5. $\left(a^b\right)^c = a^{bc}$ *for all* $a \neq 0$*.*
6. $(a \cdot b)^c = a^c \cdot b^c$ *for all* $a \neq 0$*.*
7. $a^{\frac{b}{c}} = \sqrt[c]{a^b}$ *for all* $a, b, c \neq 0$*,* b *is an integer,* c *is a positive integer, and if* a *is negative and* b *is odd, then* c *must be odd as well.*
8. *If* $a^x = a^y$ *and* $a \neq 1$*, then* $x = y$*.*

You should have the above results memorized, as they show up frequently in competition math and you need to use them for harder problems. Try the following checkpoint to see if you can apply the laws of exponentiation.

Checkpoint 1.1.

1. Solve $4^5 \cdot 8^3 = 2^x$.
2. Evaluate $243^{\frac{3}{5}}$.
3. Find x and y if $12^5 \cdot 8^3 = 2^x * 3^y$.
4. Evaluate $0.5^{-6} + 10^{-3}$.
5. Solve $\sqrt{4\sqrt{8\sqrt{2^x}}} = 128$.
6. Find the sum of the solutions of $5^{x^2} = 25^{4x+3}$.
7. Find the solutions of x to $2^x \cdot 2^{2x} \cdot 2^{3x} = (2^x)^{2x})^{3x}$.

2 Logarithm Basics and Properties

In this section, we will introduce logarithms. We define them as follows.

Definition. Let a, b, c satisfy $a^c = b$, $a, b > 0$, and $a \neq 1$. Then, we define $\log_a b = c$. This can be read as "the logarithm of b with respect to the base a is equal to c".

From this, we can see that logarithms are essentially the inverse of exponentiation. This will be further explored later in the chapter, where we consider the graphs of exponential and logarithmic functions. Next, lets discuss the several properties of logs.

Theorem 2.1. *(Law of Logarithms) The following properties govern arithmetic with logs.*

1. $a^{\log_a b} = b$.
2. $\log_a b + \log_a c = log_a(bc)$.
3. $\log_a b - log_a c = log_a \frac{b}{c}$.
4. $\log_c \left(a^b\right) = b \log_c a$.
5. $\log_c \sqrt[b]{a} = \frac{\log_c a}{b}$.
6. $\log_b a = \frac{1}{\log_a b}$.
7. $\log_b a \log_c b = \log_c a$.
8. *(Change of Base Property)* $\log_b a = \frac{\log_c a}{\log_c b}$.
9. $\log_a b \log_b c = \log_a c$.

Proof.

1. Let $c = \log_a b$. Then from the definition of logarithms, $a^c = b$. Plugging $c = \log_a b$ back into this, we have the result.

2. First we start with the equation

$$a^{\log_a b} \cdot a^{\log_a c} = a^{\log_a (bc)}.$$

This equation follows easily from Theorem 4.2.1. Now, Theorem 4.1.4, we can simplify the left-hand side to get the equation

$$a^{\log_a b + \log_a c} = a^{\log_a (bc)}.$$

Now, we can take the logarithm base a of both sides of the equation to get the desired result.

3. This result follows from substituting $\frac{1}{c}$ in for c in Theorem 4.2.2 and by noting $\log_a \frac{1}{c} = -\log_a c$, which follows from Theorem 4.2.4 by setting $b = -1$.

4. We start of with the definition of exponentiation: $a^b = a \times a \times \cdots \times a$, where there are $b - 1$ multiplication signs. Now, we can take the log base c of both sides to get

$$\begin{aligned} \log_c \left(a^b\right) &= \log_c (a \times a \times \cdots \times a) \\ &= \log_c a + \log_c a + \cdots + log_c a \\ &= b \log_c a, \end{aligned}$$

where we obtained the second equality using Theorem 4.2.2.

5. This result follows from substituting $\frac{1}{b}$ in for b in Theorem 4.2.4 and then applying Theorem 4.1.7 with $b = 1$.

6. The result follows by setting $c = a$ in Theorem 4.2.8.

7. Let $x = \log_b a$. Then $b^x = a$, so $b = a^{\frac{1}{x}}$

$$\begin{aligned} \log_c b &= \log_c \left(a^{\frac{1}{x}}\right) \\ &= \frac{1}{x} \log_c a. \end{aligned}$$

Thus,

$$\begin{aligned} \log_b a \log_c b &= x \cdot \frac{1}{x} \log_c a \\ &= \log_c a, \end{aligned}$$

as desired.

8. The result follows by dividing both sides of Theorem 4.2.7 by $\log_c b$.

□

Be careful when working with logs, as the following equations are commonly mistaken and are NOT true in general.

$$\log_c(a+b) \neq \log_c a + \log_c b$$
$$\log_c(ab) \neq \log_c b \cdot \log_c a$$

Try to solve the following problems with the logarithmic rules you just learned.

Checkpoint 2.1.

1. Evaluate $\log_4 8 + \log_8 4$.
2. Without a calculator, estimate $\log_{1.1} 96$ to the nearest whole number, given that $\log_{1.1} 2 = 7.273$ and $\log_{1.1} 3 = 11.527$.
3. How many real numbers x are there such that $\log_2 x = x$?

Checkpoint 2.2. Show the following:

- $a^{\log_b c} = c^{\log_b a}$
- $\log_{b^c} a^c = \log_b a$
- $\log_a b \log_c d = \log_c b \log_a d$.

It's worth mentioning that in many scenarios, the base of a logarithm is not specified. In this case, it is assumed that the base is 10, a standard base that's chosen to align with our numeric system. Also, when you input log on a calculator, the function is the logarithm function with a base of 10. If you want to calculate a log with a different base on a calculator, then you can use the change of base formula to get everything in logs with base-10. For example, if you wanted to calculate $\log_2 5$, you would plug in $\frac{\log 5}{\log 2}$ on your calculator.

Furthermore, the notation $\ln(x)$ is shorthand notation for $\log_e x$, where $e \approx 2.71828$ is a mathematical constant. Why this constant is important is due to calculus: the exponential function e^x is equal to its own derivative. A final note is that a log that has no defined base, for example $\log(6)$, means that the log has a base of 10 in competition math. However, in more advanced math, a log without a base usually means a natural logarithm.

3 Exponential and Logarithmic Functions

In this section, we will consider the exponential and logarithmic functions a^x and $\log_a x$ for some constant $a > 0$. We will analyze each of these graphs and discuss their relationship and transformations.

First, let's analyze the function $f(x) = a^x$. We do this by analyzing three different scenarios on our value a.

- ($a > 1$) In this scenario, we see that it is a strictly increasing function, with the function going off to infinity as $x \to +\infty$. Also, notice that the y-intercept is 1 (i.e. $a^0 = 1$). Furthermore, as $x \to -\infty$, the function goes to zero (as $f(x) = a^x = \frac{1}{a^{|x|}} \to \frac{1}{\infty} = 0$). Thus, $y = 0$ is a horizontal asymptote. Furthermore, the higher the value of a, the faster the function grows, to the right, from the point $(0, 1)$.

- If $a = 1$, then the function is simply the horizontal line $y = 1$. Boring!

- If $0 < a < 1$, the function is strictly decreasing. To see this, note that $y = a^x$ is the reflection of $y = \left(\frac{1}{a}\right)^x$ across the y-axis. This is because

$$\left(\frac{1}{a}\right)^x = \left(a^{-1}\right)^x = a^{-x}.$$

  From this, we can also see that the intersection of $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$ occurs precisely at the point $(0, 1)$.

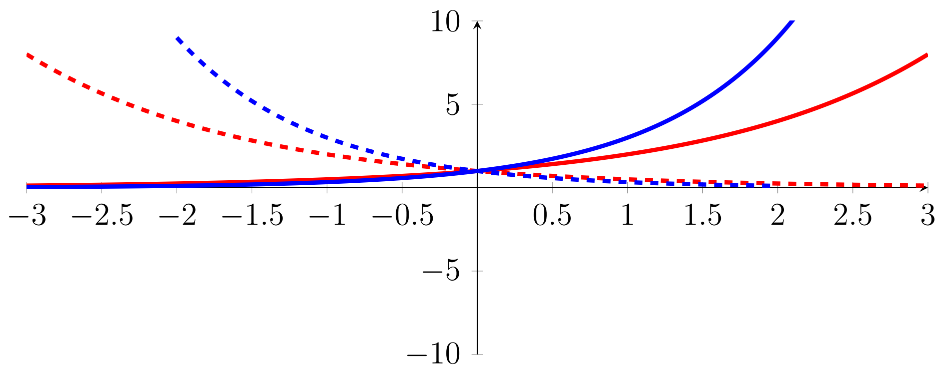

Figure 1: The figure shows the plots of $y = 2^x$ (in solid red), $y = \left(\frac{1}{2}\right)^x$ (in dashed red), $y = 3^x$ (in solid blue), and $y = \left(\frac{1}{3}\right)^x$ (in dashed blue) on the interval $[-3, 3]$. This plot illustrates the observations made above on the exponential function.

Next, let's analyze the function $f(x) = \log_a x$ for different values of a.

- When $a > 1$, the function $y = \log_a x$ is an increasing function that has an asymptote at $x = 0$ (approaches $-\infty$ as $x \to 0^+$), and increases off to infinity as $x \to +\infty$. Additionally, notice that the x-intercept is $(1, 0)$.

- When $a = 1$, $y = \log_a x$ is undefined for all $x \neq 1$, so this value of a isn't useful at all!

- When $a < 1$, the function is in fact strictly decreasing and also has an x-intercept $(1, 0)$. It's easy to see that this function is just the reflection of $\log_{\frac{1}{a}} x$ across the x-axis by the change-of-base identity:

$$\begin{aligned}\log_{\frac{1}{a}} x &= \frac{\log_a x}{\log_a \frac{1}{a}} \\ &= -\log_a x.\end{aligned}$$

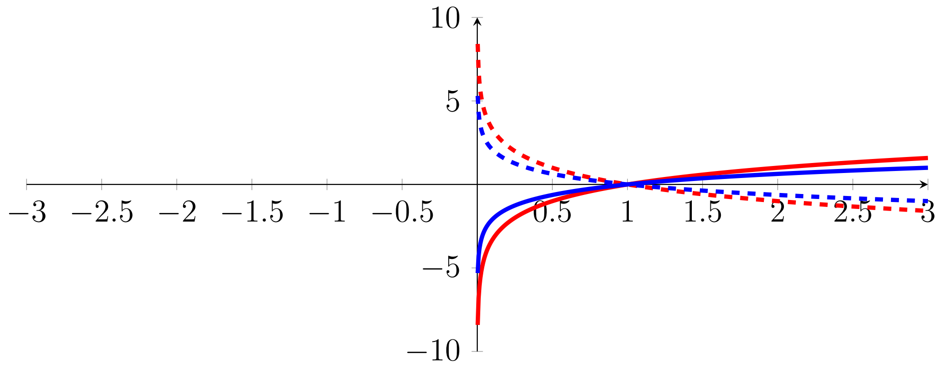

Figure 2: The figure shows the plots of $y = \log_2 x$ (in solid red), $y = \log_{\frac{1}{2}} x$ (in dashed red), $y = \log_3 x$ (in solid blue), and $y = \log_{\frac{1}{3}} x$ (in dashed blue) on the interval $[-3, 3]$. This plot illustrates the observations made above on the log function.

From our above discussion and graphs of various exponential and logarithmic functions above, it should've become clear that the exponential function is in fact the inverse of the logarithmic function. This can be easily verified algebraically, but also can be seen graphically by noting that reflecting the exponential function across $y = x$ gives the corresponding logarithmic function.

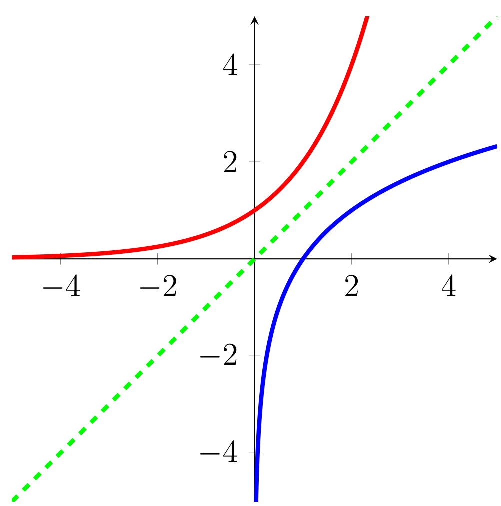

Figure 3: The figure shows the plots of $y = 2^x$ (in solid red) and $y = \log_2 x$ (in solid blue) on the interval $[-4, 4]$. This plot illustrates the observations made above that the logarithmic function and the exponential function are inverses of each other.

4 Problem Solving Strategies with Logs and Exponents

In this section, we will introduce a few problems to help illustrate some basic problem solving strategies regarding logarithms that are commonly used in competition math.

The identities in Theorem 4.2 are the primary tool used to solve equations involving logarithms. However, there are many other techniques that are useful to solving these problems. These include raising both sides of the equation to a certain power, eliminating some logarithms, and converting a problem to solving an exponential equation; this turns out to make the problem straightforward in many cases. Many of the exercises in this chapter will help you hone your problem-solving skills using logarithms and exponents, as it takes practice to know which technique/identity should be applied in different scenarios. Here are a few examples that will help you get into such problem solving techniques.

Example 4.1. Suppose that

$$\log_{10} xy^3 = 1$$
$$\log_{10} x^2 y = 1$$

What is $\log_{10} xy$? *Source: AMC*

Solution. Let's try to remove the logs from the equation using exponentiation, as logs are more complicated to deal with than exponents. This can be done by rewriting these equations into exponent form, giving us

$$10 = xy^3 \tag{1}$$
$$10 = x^2 y \tag{2}$$

Now, how do we find xy from these equations? In order to find xy, we will need to multiply the equations in a way such that the exponent of x equals the exponent of y. This can be done by multiplying the square of (4.2) and (4.1) to get $x^5y^5 = 10^3$. We can now convert this back to log form which results in $\log_{10} x^5y^5 = 3$. Furthermore, using logarithm identities, we have

$$\begin{aligned} \log_{10} x^5y^5 &= \log_{10}(xy)^5 \\ &= 5\log_{10} xy. \end{aligned}$$

Therefore, $\log_{10} xy = \frac{3}{5}$. △

In this problem, we showed you how to convert a log equation to an exponential equation. This is very important when it comes to problem solving because logs are harder to work with then exponents.

Another important strategy to use when dealing with log expressions or equations is substitution. In this strategy, we can substitute commonly occurring log expressions with variables to make the problem easier. This technique is shown in the following example.

Example 4.2. Find $\log_{10} \frac{x}{y}$ if the following equations are true with $x > y$:

$$\log_{10} x + \log_{10} y = 10$$
$$\log_{10} x \cdot \log_{10} y = 16$$

Solution. Solving this problem seems messy at first, especially considering what the problem asks us to find! Let's try to simplify the problem by removing the logs. We can do this by substituting $a = \log_{10} x$ and $b = \log_{10} y$. We can then rewrite the two equations as $a + b = 10$ and $ab = 16$. Now, lets take a look at what the problem asks for. It asks for $\log_{10} \frac{x}{y}$, which we can rewritten as $\log_{10} x - \log_{10} = a - b$ using the subtraction identity for logs. With this information, we now just need to find $a - b$. We can square both sides of the equation $a + b = 10$ to get $a^2 + 2ab + b^2 = 100$. Since $ab = 16$, we can subtract $4ab = 64$ from both sides to get $a^2 - 2ab + b^2 = 36$. Now we can conveniently factor the left-hand side to get $(a - b)^2 = 36$. Since $x > y$, we have that $a > b$, so we take the positive solution of $a - b$. Hence, $a - b = \log_{10} \frac{x}{y} = 6$. △

With the correct substitution of logs, this problem solving strategy can be extremely useful in contests. As long as you know your identities, you can turn a complex log problem into a simpler algebra problem.

Next, let's take a look at a complex exponent equation that is similar to problems in past math contests.

Example 4.3. Find the number of solutions x to the equation

$$\left(x^2 - 11x + 29\right)^{2x^2 - 9x - 18} = 1.$$

Solution. In order to solve this, we first need to consider how to get the left hand side to equal 1. Overall, there are three ways for this to happen. The first case is when the base equals 1. If the base is 1, then the exponent can be any real number. In this case, there are 2 solutions satisfying $x^2 - 11x + 29 = 1$ (namely $x = 4, 7$). The second case is when the exponent equals 0. There are 2 solutions to $2x^2 - 9x - 18 = 0$ (namely $x = -\frac{3}{2}, 6$. The last (and commonly forgotten!) case is when the base is -1 and the exponent is an even integer. There are two solutions to $x^2 - 11x + 29 = -1$: namely $x = 5, 6$. We can plug in both numbers in the exponent to find that the exponent is odd when $x = 5$ and even when $x = 6$. Therefore, we only have one solution to this case: $x = 6$. After a quick check that our solutions to each of the cases don't coincide (a common pitfall!), we get that there are 5 solutions. △

Another thing to be careful is when dealing with even exponents in equations. This includes squaring an equation with a square root. As two number numbers raised to an even power can equal the same number, this can produce multiple or extraneous solutions. For example, $x^4 = 81$ has 2 real solutions, -3 and 3.

Finally, we'll take a look at solving a system of log equations. Like solving other systems of nonlinear equations, we aim to substitute in variables to make the equations nicer. This aids us in eliminating variables, which in turn gives us the solution to the system.

Example 4.4.

Find both ordered triplets (x, y, z) that satisfy the system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0. \end{aligned}$$

Source: AIME

Solution. Using the sum of logarithms identity, let's rewrite the equations in the system as follows:

$$\begin{aligned} 3 + \log_{10} 2 + \log_{10} x + \log_{10} y - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10} 2 + \log_{10} y + \log_{10} z - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10} z + \log_{10} x - (\log_{10} z)(\log_{10} x) &= 0. \end{aligned}$$

Now, let's get rid of the logs by making the substitutions $a = \log_{10} x$, $b = \log_{10} y$, and $c = \log_{10} z$. Doing these substitutions and doing a bit of rearrangement and factoring (Simon's Favorite Factoring Trick!) we get

$$\begin{aligned} (a-1)(b-1) &= \log_{10} 2 \\ (b-1)(c-1) &= \log_{10} 2 \\ (c-1)(a-1) &= 1. \end{aligned}$$

Now, we multiply all three of the equations and take the square root to give us

$$(a-1)(b-1)(c-1) = \pm \log_{10} 2.$$

Dividing this by each equation in the previous system, we get $c - 1 = 1$, $a - 1 = 1$, and $b - 1 = \log_{10} 2$, or $c - 1 = -1$, $a - 1 = -1$, and $b - 1 = -\log_{10} 2$. Thus our two solutions are $(a, b, c) = (2, \log_{10} 2 + 1, 2), (0, \log_{10} 2 + 1, 0)$. Writing each in terms of either x, y, or z from our original definitions and using the simplifications $10^{(\log_{10} 2)+1} = 20$ and $10^{(-\log_{10} 2)+1} = 5$ (from the basic logarithm and exponent properties) we get the two solutions $(x, y, z) = \boxed{(100, 20, 100), (1, 5, 1)}$. △

We hope these examples have given you a good introduction into using problem strategies to solve log problems. However, the only way to get a good grasp of them is to solve many problems involving logs, which includes the exercises here. Try them out!

5 Exercises

1. ⋆ What is the value of a for which $\frac{1}{\log_2 a} + \frac{1}{\log_3 a} + \frac{1}{\log_4 a} = 1$? *Source: AMC*

2. ⋆ For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$? *Source: AMC*

3. $\star$ For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40?$$

Source: AMC

4. $\star$ The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b? *Source: AMC*

5. $\star$ Let x, y and z all exceed 1 and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$ and $\log_{xyz} w = 12$. Find $\log_z w$. *Source: AIME*

6. $\star$ For what value of x does $10^x \cdot 100^{2x} = 1000^5$? *Source: AMC*

7. $\star$ What is the value of the expression

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!},$$

where

$$100! = 100 \cdot 99 \cdot 98 \cdot 97 \cdot \cdots \cdot 2 \cdot 1?$$

Source: AHSME

8. $\star\star$ Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$. *Source: AIME*

9. $\star\star$ The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$
$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$. *Source: AIME*

10. $\star\star$ Let $N = 2^{(2^2)}$ and x be a real number such that $N^{\left(N^N\right)} = 2^{(2^x)}$. Find x. *Source: HMMT*

11. $\star\star$ What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

Source: AMC

12. $\star\star$ The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n. *Source: AIME*

13. $\star\star$ The graphs of $y = \log_3 x, y = \log_x 3, y = \log_{\frac{1}{3}} x$, and $y = \log_x \dfrac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive x-coordinates lie on two or more of the graphs? *Source: AMC*

14. $\star\star$ The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of what length? *Source: AMC*

15. $\star\star\star$ The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r). *Source: AIME*

16. $\star\star\star$ Compute the positive real number x satisfying

$$x^{\left(2x^6\right)} = 3.$$

Source: HMMT

17. $\star\star\star$ Let $f(x) = (x^2 + 3x + 2)^{\cos(\pi x)}$. Find the sum of all positive integers n for which

$$\left|\sum_{k=1}^{n} \log_{10} f(k)\right| = 1.$$

Source: AIME

18. $\star\star\star$ In a Martian civilization, all logarithms whose bases are not specified as assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$

$$\log_{\log x}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b. *Source: AIME*

19. $\star\star\star$ There are positive integers x and y that satisfy the system of equations

$$\begin{aligned}\log_{10} x + 2\log_{10}(\gcd(x, y)) &= 60\\ \log_{10} y + 2\log_{10}(\text{lcm}(x, y)) &= 570.\end{aligned}$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$. *Source: AIME*

20. $\star\star\star$ Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$? *Source: AMC*