Week 4:
Inequalities

An inequality is just an equation with the equality symbol replaced with one of the for inequality symbols: $<$, $\leq$, $>$, and $\geq$. Recall that $<$ and $>$ mean less than and greater than, and $\leq$ and $\geq$ means less than or equal to and greater than or equal to, respectively.

First, we will consider the operation of adding/subtracting a constant on both sides of an inequality, and multiplying/dividing an equality by a constant.

Theorem 2.1. *We have the following if $a \geq b$ and c is some constant.*

1. $a + c \geq b + c$
2. $a - c \geq b - c$
3. $a \cdot c \geq b \cdot c$ *if* $c \geq 0$*; otherwise,* $a \cdot c \leq b \cdot c$
4. *For* $c \neq 0$*: if* $c > 0$*, then* $\frac{a}{c} \geq \frac{b}{c}$*; otherwise,* $\frac{a}{c} \leq \frac{b}{c}$

Theorem 2.2. *Suppose a, b, c, and d are real numbers. Then if $a \geq b$ and $c \geq d$, we can write $a + c \geq b + d$.*

Theorem 1.3. *Suppose a, b, c, and d are real numbers such that b and c are non-negative. Then if $a \geq b$ and $c \geq d$, we can write $ac \geq bd$. However, if we remove the condition that b and c are non-negative, then the statement doesn't hold.*

Example 2.1. Let $a = 6$, $b = 4$, $c = 2$, $d = -1$, and $e = -4$, so that $a \geq b \geq c \geq d \geq e$. Which of the following statements are true? (Try to do this first by considering the sign of the variables, so that you can easily generalize these examples. You can always check your answers by plugging the numbers in.)

1. $a + c \geq b + c$
2. $a - d \geq b - d$
3. $a - d \geq b - e$
4. $a \cdot c \geq b \cdot c$
5. $\frac{a}{d} \geq \frac{b}{d}$
6. $\frac{a}{e^2} \geq \frac{b}{e^2}$
7. $a + d \geq b + e$
8. $\frac{a}{d} > \frac{b}{e}$
9. $ad \geq be$

Checkpoint 1.1. Show that for all real numbers a, b, c, and d such that $a \geq b$ and $c \geq d$, $ac + bd \geq ad + bc$.

2.2 The Trivial Inequality

The trivial inequality states that the square of any real number is non-negative. For example, $x^2 \geq 0$, $(a+b)^2 \geq 0$, and $(a-b)^2 \geq 0$ are all examples of the trivial inequality. This might seem very simple, but it is the basis of many inequalities.

Example 2.2. Solve this equation over the reals: $(x^2 - 6x + 15)(9y^2 + 12y + 12) = 48$

Example 2.3. Show that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for all reals x, y, and z.

Checkpoint 2.1. Show that for all real numbers x and y,

$$5x^2 + 2xy + 5y^2 \geq 0.$$

2.3 The AM-GM Inequality

The AM-GM Inequality is a powerful inequality that holds for any set of non-negative real numbers. "AM" stands for arithmetic mean (or the average) and "GM" stands for geometric mean (the nth root of the product of the numbers, if we are dealing with n numbers) in the name of the inequality. We will first present the AM-GM result for two non-negative variables.

Theorem 2.4. *For any two non-negative real numbers a and b, we have that*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Theorem 2.5. *(AM-GM) If a_1, $a_2, \dots,$ a_n non-negative real numbers, then*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

We have equality if and only if $a_1 = a_2 \ldots = a_n \geq 0$

Example 2.4. What is the minimum value of $x + \frac{1}{x}$, where x is a positive number? Prove this value.

Checkpoint 3.1. Find the minimum value of $4x + \frac{9}{x}$ where x is a positive number, and find the x that produces this number.

Example 2.5. Find the minimum value of $\frac{9x^2+36x+52}{12(x+2)}$ given that x is a positive integer.

Checkpoint 3.2. Find the minimum value of

$$f(x) = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}}$$

for $x > 0$. *Source: AoPS*

2.4 The Cauchy-Schwarz Inequality

Next, we will discuss a very important inequality. It doesn't show up that frequently in competition math, but it can be converted into vector form to give an extremely powerful result that is used in a wide range of advanced mathematics. Here, we'll discuss a form of the inequality that doesn't involve vectors.

Theorem 2.6. *(Cauchy-Schwarz) For real numbers a_i and b_i, we have the following:*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Equivalently,

$$(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$$

Theorem 2.7. *Equality is achieved in the Cauchy-Schwarz inequality if and only if there exists some real number t such that $a_i = tb_i$ for $i = 1, 2 \dots n$. In other words, when we take each a_i and divide by corresponding b_i, we should get the same value each time.*

Example 2.6. Let x, y, and z be real numbers such that $x^2 + y^2 + z^2 = 1$. Find the maximum value of $3x + 4y + 12z$.

Example 2.7. Find the minimum value of the expression $(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$ over the reals.

Checkpoint 4.1. Show that if we have $a_i = tb_i$ for some real number t, then equality is achieved in the Cauchy-Schwarz inequality.

Checkpoint 4.2. Let a and b be real numbers. Find the maximum value of $a\cos\theta + b\sin\theta$ in terms of a and b. *Source: AoPS*

Checkpoint 4.3. Let a, b, and t be real numbers such that $a + b = t$. Find, in terms of t, the minimum value of $a^2 + b^2$. *Source: AoPS*

2.5 Geometric Inequalities

Inequalities don't have to just be algebraic. There are also geometric inequalities, the most important one being the triangle inequality, as we'll see in the following theorem. In this section, we will discuss the application of these inequalities in geometry problems. We will omit the discussion of vectors, but if you are familiar with them, we encourage you to try to work with and prove the inequalities in this section using vectors.

Theorem 2.8. *For any triangle, we have that any of the side lengths is less than the sum of the other two. In particular, for a triangle with side lengths a, b, and c with $a \leq b \leq c$, we have $c < b + a$.*

Example 2.8. Find the sum of the smallest possible value and largest possible value of integer x if a triangle with sides 5, 7, and x exist.

We can also extend the triangle inequality to polygons.

Theorem 2.9. *For any polygon, any side length is less than the sum of the other sides. Furthermore, the non-negative real numbers* $s_1, s_2 \dots s_n$ *can be the sides of a polygon if and only if for* $i = 1, 2 \dots n$, $s_i < \sum_{j=1, j \neq i}^{n} s_j$.

Theorem 2.10. *Given a triangle ΔABC with side lengths a, b and c with $a \leq b \leq c$, we have the following:*

1. $a^2 + b^2 < c^2 \Leftrightarrow \Delta ABC$ *is obtuse.*

2. $a^2 + b^2 = c^2 \Leftrightarrow \Delta ABC$ *is right.*

3. $a^2 + b^2 > c^2 \Leftrightarrow \Delta ABC$ *is acute.*

Checkpoint 5.1. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod? *S*ource: AMC 10