Online Summer Math Circle

Week 3: Algebra Techniques

Solving Systems of Equations

Example 10.1. If A, B, C, D, E, F and G satisfy the equations shown, what is the value of each variable?

$$\begin{aligned}
A + B + C &= 5 \\
B + C + D &= 7 \\
C + D + E &= 9 \\
D + E + F &= 11 \\
E + F + G &= 13 \\
F + G &= 10 \\
A + F + G &= 11
\end{aligned}$$

Example 10.2. Find the sum of of the 5 variables if the following equations are true:

A + B + C + D + E

$$A + B + C + D + 2E = 10$$
$$A + B + C + 2D + E = 9$$
$$A + B + 2C + D + E = 8$$
$$A + 2B + C + D + E = 7$$
$$2A + B + C + D + E = 8$$

$$6A + 6B + 6C + 6D + 6E = 42$$

$$A + B + C + D + E = \boxed{7}$$

Example 10.3. Find all ordered triples (a, b, c) such that

$$ab = 2$$
$$bc = 3$$
$$ac = 6.$$

$$a^2 b^2 c^2 = 6^2$$

$$abc = 6 \quad \text{or} \quad abc = -6$$

$c = 3 \quad a = 2$

$b = 1$

$(2, 1, 3)$

$c = -3 \quad a = -2$

$b = -1$

$(-2, -1, -3)$

Checkpoint 10.1. Find $a+b$ (without explicitly finding a and b) if a and b satisfy $a+2b=20$ and $4a+3b=15$. After doing this, solve explicitly for a and b.

$$5a+5b=35$$

$$\boxed{a+b=7}$$

$$\boxed{b=13}$$

$$a+26=20$$

$$\boxed{a=-6}$$

Special Factorizations

1. $\star$ $x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$ In particular,

   (a) $x^2 - y^2 = (x+y)(x-y)$

   (b) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

   (c) $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1)$

2. $\star$ If n is odd,

$$x^n + y^n = (x+y)\left(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}\right)$$

   In particular,

   (a) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

   (b) $x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + (-1)^{n-1})$

Special Factorizations (cont.)

3. $\star$ The following factorizations can help you find the sum of powers of roots.

   (a) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

   (b) $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

   For example, if $r_1, r_2, \ldots, r_k$ are the roots of a polynomial, we can find $r_1^n + r_2^n + \ldots + r_k^n$ by using Vieta's formulas and the factorizations above. This technique is known as Newton's Sums. Note that the above factorizations can be generalized, but we omit the general factorization since it doesn't show up that often competition math and it can be easily derived.

4. $\star$ (Simon's Favorite Factoring Trick, or SFFT for short) $xy + ax + by + ab = (x+b)(y+a)$. While this is simply foiling, it's important that you look out for expressions of this form. In most situations, x and y are the variables, and a and b are the constants.

Special Factorizations (cont.)

5. Sophie Germain's: $a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$.

6. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{1}{2}(a + b + c)\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right]$.

   (a) $a^3 + b^3 + c^3 = 3abc$ if and only if $a + b + c = 0$ or $a = b = c$.

   (b) If you have the terms a^3, b^3, and ab, then you can factorize it in this manner while allowing $c = \pm 1$.

7. (Lagrange) The abundance of squares formula:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2.$$

Example 10.6. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.
Source: AIME

$$3x^2y^2 + y^2 - 30x^2 = 517$$

$$(3x^2+1)(y^2-10) = 517-10$$

$$(3x^2+1)(y^2-10) = 507$$

$$3 \cdot 13^3$$

$$y^2-10 = 39 \qquad 3x^2+1 = 13$$

$$y^2 = 49 \qquad x^2 = 4 \qquad 4 \cdot 49 \cdot 3 =$$

$$\boxed{588}$$

Example 10.7. Compute the value of

$$\frac{2014^4 + 4 \cdot 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \cdot 2013^4}{2013^2 + 4025^2}.$$

Source: BMO $a = 2012$, $b = 2013$, $c = 2014$

$$\frac{c^4 + 4b^4}{b^2 + (b+c)^2} - \frac{a^4 + 4b^4}{b^2 + (a+b)^2}$$

$$\frac{(c^2-2bc+2b^2)\cancel{(a^2+2bc+2b^2)}}{\cancel{c^2+2bc+2b^2}}$$

$$(c^2-2bc+\cancel{2b^2})-(a^2-2ab+\cancel{2b^2})$$

$$c^2-a^2-2bc+2ab$$

$$(c-a)(c+a)-2b(c-a)$$

$$\cancel{2\cdot 4026}-\cancel{2\cdot 2013}-2=\boxed{0}$$

Example 10.8. Let a, b, and c be real numbers such that $a+b+c=1$. Show that

$$a^3+b^3+c^3-1=3(abc-ab-bc-ca).$$

Source: AwesomeMath

$$a^3+b^3+c^3-3abc=\cancel{(a+b+c)}(a^2+b^2+c^2-ab-bc-ca)$$

$$1=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$a^3+b^3+c^3-1=3abc-3(ab+bc+ca)$$

$$a^3+b^3+c^3-1=3(abc-ab-bc-ca)$$

$\square$

Checkpoint 10.2. Solve the system of equations

$$\begin{cases} x + y = xy - 11 \\ y + z = yz - 19 \\ z + x = zx - 14 \end{cases}$$

SFFT

$$xy - x - y + 1 = 11 + 1$$

$$(x-1)(y-1) = 12$$

$$(y-1)(z-1) = 20$$

$$(z-1)(x-1) = 15$$

$$\boxed{x = 4,\ y = 5 \quad z = 6}$$

$$\boxed{x = -2,\ y = -3 \quad z = -4}$$

Radicals and Conjugates

$\overline{x+y} = x - y$

Checkpoint 10.2. After rationalizing the numerator of $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}$, find the denominator in simplest form. *Source: AHSME*

$$(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$$

$$\frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}{\sqrt{3}\ (\sqrt{3}+\sqrt{2})} = \frac{3-2}{\sqrt{3}(\sqrt{3}+\sqrt{2})}$$

$$\boxed{3+\sqrt{6}}$$

Breaking up Nested Radicals

Example 10.7. De-nest $\sqrt{10+2\sqrt{21}}$.

$$\sqrt{a}+\sqrt{b} = \sqrt{10+2\sqrt{21}}$$

$$a+b+2\sqrt{ab} = 10+2\sqrt{21}$$

$$a+b=10 \qquad ab=21$$

$$a=7,\ b=3 \implies \boxed{\sqrt{7}+\sqrt{3}}$$

Checkpoint 10.3. De-nest $\sqrt{11 + 2\sqrt{30}}$.

$$\sqrt{5} + \sqrt{6}$$

Theorem 10.1. *The following holds:*

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Clever Substitutions

Example 10.8. What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$? *Source: AIME*

$$y = x^2 + 18x + 30$$

$$y = 2\sqrt{y+15}$$

$$y^2 = 4y + 60$$

$$y^2 - 4y - 60 = 0$$

$$(y-10)(y+6) = 0$$

$$y = 10 \qquad \cancel{y = -6}$$

$$10 = x^2 + 18x + 30$$

$$x^2 + 18x + 20 = 0$$

$$\boxed{20}$$

Example 10.9. Find all real numbers a, b, and c such that

$$\sqrt[3]{a-b} + \sqrt[3]{b-c} + \sqrt[3]{c-a} = 0$$

$x = \sqrt[3]{a-b}$ $y = \sqrt[3]{b-c}$ $z = \sqrt[3]{c-a}$

$x + y + z = 0$

$x^3 + y^3 + z^3 - 3xyz = \underbrace{\cancel{(x+y+z)}(\ldots)}_{=0}$

$x^3 + y^3 + z^3 = 3xyz$

$a - b + b - c + c - a = 0$ $\cancel{x = 0 \Rightarrow a - b = 0, a = b}$

$3xyz = 0$ $\boxed{\text{all } (a, b, c) \text{ s.t. } a = b, b = c, \text{ or } c = a}$

Checkpoint 10.4. Let $S = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$. Then S equals:

(A) $(x-2)^4$ **(B)** $(x-1)^4$ **(C)** x^4 **(D)** $(x+1)^4$ **(E)** x^4+1

Source: AHSME

$y = x - 1$

$S = y^4 + 4y^3 + 6y^2 + 4y + 1$

$= (y+1)^4 = x^4$ $\boxed{C}$

Symmetry

Example 10.9. If $x + \frac{1}{x} = 5$, then what is $x^2 + \frac{1}{x^2}$?

Example 10.10. Find all the solutions to the following equation: $x^4 + 3x^3 - 8x^2 + 3x + 1 = 0$.

1. Divide the polynomial by the middle term (ignoring the coefficient). For example, divide the polynomial $x^5 + 4x^2 - 3x^3 + 4x + 1$ by x^3, and the polynomial $x^6 - 3x^5 + x^3 + 3x + 1$ by x^3. More generally, if your polynomial has degree n, divide by $x^{\lceil \frac{n}{2} \rceil}$.

2. Let $y = x + \frac{1}{x}$, and write the expression in terms of y by writing $x + \frac{1}{x}$, $x^2 + \frac{1}{x^2} \ldots$ $x^{\lceil \frac{n}{2} \rceil} + \frac{1}{x^{\lceil \frac{n}{2} \rceil}}$ in terms of y.

3. You should now have a polynomial in y. Find the roots of this polynomial.

4. Now, solve for x by setting $x + \frac{1}{x}$ to all the solutions you got for y. For each value of y, multiply both sides by x to get a quadratic in x. From here, you can either factor the expression or use the quadratic formula to solve.

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

$$x^3 + \frac{1}{x^3} = y^3 - 3y$$

$$x^4 + \frac{1}{x^4} = y^4 - 4y^2 + 2$$

Example 10.12. In the following system of equations, solve for the product xyz.

$$x + \frac{1}{y} = 4$$
$$y + \frac{1}{z} = 1$$
$$z + \frac{1}{x} = \frac{7}{3}$$

Checkpoint 10.6. In the following system of equations, solve for the product xyz in terms of a, b, and c.

$$x + \frac{1}{y} = a$$
$$y + \frac{1}{z} = b$$
$$z + \frac{1}{x} = c$$