Iowa City Math Circle Handouts
Inequalities

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1 Arithmetic with Inequalities

An inequality is just an equation with the equality symbol replaced with one of the for inequality symbols: $<$, $\leq$, $>$, and $\geq$. Recall that $<$ and $>$ mean less than and greater than, and $\leq$ and $\geq$ means less than or equal to and greater than or equal to, respectively.

First, we will consider the operation of adding/subtracting a constant on both sides of an inequality, and multiplying/dividing an equality by a constant.

Theorem 1.1. *We have the following if $a \geq b$ and c is some constant.*

1. $a + c \geq b + c$
2. $a - c \geq b - c$
3. $a \cdot c \geq b \cdot c$ *if* $c \geq 0$*; otherwise,* $a \cdot c \leq b \cdot c$
4. *For* $c \neq 0$*: if* $c > 0$*, then* $\frac{a}{c} \geq \frac{b}{c}$*; otherwise,* $\frac{a}{c} \leq \frac{b}{c}$

We can also perform simple arithmetic with two inequalities. We can add any two inequalities and multiply two inequalities with certain constraints. However, subtraction and division with inequalities don't work as you might expect. For example, $8 > 4$ and $6 > 1$, but $\frac{8}{6} < \frac{4}{1}$ and $8 - 6 < 4 - 1$. Addition and multiplication of inequalities will be summarized in the following theorems.

Theorem 1.2. *Suppose a, b, c, and d are real numbers. Then if $a \geq b$ and $c \geq d$, we can write $a + c \geq b + d$.*

Proof. From the two given inequalities, we can deduce $a - b \geq 0$ and $c - d \geq 0$ by Theorem 2.1. Adding $c - d$ to both sides of the first inequality, we have $a - b + c - d \geq c - d \geq 0$. Rearranging gives us the desired inequality. □

Theorem 1.3. *Suppose a, b, c, and d are real numbers such that b and c are non-negative. Then if $a \geq b$ and $c \geq d$, we can write $ac \geq bd$ if However, if we remove the condition that b and c are non-negative, then the statement doesn't hold.*

Proof. First, suppose a, b, c, and d are real numbers such that b and c are non-negative. Then we can multiply both sides of the equation $c \geq d$ by b to get $bc \geq bd$. Similarly, we can multiply both sides of the equation $a \geq b$ by c to get $ac \geq bc$. Thus,

$$ac \geq bc \geq bd,$$

as desired.

From our above chain of inequalities, we see that the statement doesn't hold if b and c are negative since when we multiply the two inequalities by them, the signs of the inequalities get flipped. Furthermore, consider the ordered pair $(a, b, c, d) = (-1, -2, -3, -4)$. Clearly $a \geq b$ and $c \geq d$; however, $ac = 3$ and $bd = 8$, so $ac < bd$. □

You should become quite familiar with the above laws of inequalities. The following example should give you an idea of these laws and common pitfalls.

Example 1.1. Let $a = 6$, $b = 4$, $c = 2$, $d = -1$, and $e = -4$, so that $a \geq b \geq c \geq d \geq e$. Which of the following statements are true? (Try to do this first by considering the sign of the variables, so that you can easily generalize these examples. You can always check your answers by plugging the numbers in.)

1. $a + c \geq b + c$
2. $a - d \geq b - d$
3. $a - d \geq b - e$
4. $a \cdot c \geq b \cdot c$
5. $\frac{a}{d} \geq \frac{b}{d}$
6. $\frac{a}{e^2} \geq \frac{b}{e^2}$
7. $a + d \geq b + e$
8. $\frac{a}{d} > \frac{b}{e}$
9. $ad \geq be$

Solution. Answers:

1. True by Theorem 1.1.
2. True by Theorem 1.1.
3. False, subtraction of inequalities doesn't work!

4. True by Theorem 1.1.
5. False by Theorem 1.1. d is negative.
6. True by Theorem 1.1 since $e^2 > 0$.
7. True by Theorem 1.2.
8. False, division of inequalities doesn't work!
9. False by Theorem 1.3.

△

Checkpoint 1.1. Show that for all real numbers a, b, c, and d such that $a \geq b$ and $c \geq d$, $ac + bd \geq ad + bc$.

2 The Trivial Inequality

The trivial inequality states that the square of any real number is non-negative. For example, $x^2 \geq 0$, $(a+b)^2 \geq 0$, and $(a-b)^2 \geq 0$ are all examples of the trivial inequality. This might seem very simple, but it is the basis of many inequalities.

Example 2.1. Solve this equation over the reals: $(x^2 - 6x + 15)(9y^2 + 12y + 12) = 48$

Solution. Multiplying the two quadratics on the left hand side will get very messy, so we look for other methods to solve this problem. There are many other ways to deal on quadratics, such as completing the square. Let's try that on both quadratics. Doing this results in $((x-3)^2 + 6)((3y+2)^2 + 8) = 48$. Notice that the minimum value of the first quadratic is 6, and the minimum value of the second quadratic is 8, according to the trivial inequality. Since $6 \cdot 8 = 48$, the only way to make this equation true is to set each quadratic to its minimum value. By the trivial inequality, the minimum value of each quadratic occurs when $(x-3)^2 = 0$ and $(3y+2)^2 = 0$, or $x - 3 = 0$ and $3y + 2 = 0$, respectively. Therefore, the solution to the equation is $(x, y) = \boxed{(3, -\frac{2}{3})}$. △

Example 2.2. Show that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for all reals x, y, and z.

Solution. Bringing all the terms to the left-hand side, we obtain the expression $x^2 + y^2 + z^2 - xy - yz - zx$ on one side. The squared and cross product terms should remind you of $(x-y)^2$ (rather than $(x+y)^2$, as the cross product terms are negated). Because the inequality is symmetric in x, y,and z, we should not only look at $(x-y)^2$, but also $(y-z)^2$ and $(z-x)^2$. After expansion, we have

$$x^2 - 2xy + y^2 \geq 0,$$
$$y^2 - 2yz + z^2 \geq 0,$$
$$z^2 - 2zx + x^2 \geq 0$$

by the trivial inequality. By adding these inequalities and combining like terms, we obtain

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx \geq 0.$$

Dividing both sides by 2 and moving the cross prouduct terms to one side, we get the desired inequality: $x^2 + y^2 + z^2 \geq xy + yz + zx$. △

This inequality may seem simple in appearance but make sure you remember it and know its proof because it shows up frequently in competition math. Here is a nice problem for you to try:

Checkpoint 2.1. Show that for all real numbers x and y,

$$5x^2 + 2xy + 5y^2 \geq 0.$$

3 The AM-GM Inequality

The AM-GM Inequality is a powerful inequality that holds for any set of non-negative real numbers. "AM" stands for arithmetic mean (or the average) and "GM" stands for geometric mean (the nth root of the product of the numbers, if we are dealing with n numbers) in the name of the inequality. We will first present the AM-GM result for two non-negative variables.

Theorem 3.1. *For any two non-negative real numbers a and b, we have that*

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Hint: Use the trivial inequality!

Proof. One form of the Trivial Inequality is $(a-b)^2 \geq 0$. We can expand this out to get $a^2 - 2ab + b^2 \geq 0$. Rearranging gives us $a^2 + b^2 \geq 2ab$. Then, we can add $2ab$ to both sides and factor the left hand side to get $(a+b)^2 \geq 4ab$. With this, we can take the square root both sides without trouble because both sides are non-negative. This gives us $a + b \geq 2\sqrt{ab}$. Dividing by 2 on both sides will give us the desired inequality. □

The AM-GM inequality for two variables essentially says that the arithmetic mean is greater than or equal to the geometric mean. One can observe that by starting with the Trivial Inequality $(a-b)^2 \geq 0$, equality holds in the previous theorem if and only if $a = b$.

Next, we discuss the general AM-GM inequality. It can be summarized as follows: the average of a set of any non-negative real numbers is at least the geometric mean of the numbers.

Theorem 3.2. *(AM-GM) If a_1, $a_2, \ldots$, a_n non-negative real numbers, then*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

We have equality if and only if $a_1 = a_2 \ldots = a_n \geq 0$

We will leave the proof of this to the reader. Most proofs involve induction, but there are some other elegant proofs including a simple application of Jensen's Inequality. This is however out of the scope of this handout, so we will now focus on applying the AM-GM inequality to solve math contest problems.

Example 3.1. What is the minimum value of $x + \frac{1}{x}$, where x is a positive number? Prove this value.

Solution. We can easily use some guess and check to find that the minimum value of this expression is 2, when $x = 1$. However, let's use the AM-GM inequality to prove this answer. We can set x and $\frac{1}{x}$ as the two variables, and plugging them in results in $\frac{x+\frac{1}{x}}{2} \geq 1$. Multiplying by 2 on both sides results in $x + \frac{1}{x} \geq 2$; therefore, if we can choose x such that $x + \frac{1}{x} = 2$, then this value of x minimizes the expression. By our theorem above (addressing the equality case), we can only have $x + \frac{1}{x} = 2$ if $x = \frac{1}{x}$ for non-negative x. Solving this equation, we have $x = 1$. Hence, setting $x = 1$ minimizes the expression $x + \frac{1}{x}$ with value $\boxed{2}$.

Note that once we obtained $x + \frac{1}{x} \geq 2$, we were not done! We had to check whether 2 could be obtained, by seeing if we could have $x = \frac{1}{x}$ (in general, $a_1 = a_2 \ldots = a_n$). △

Checkpoint 3.1. Find the minimum value of $4x + \frac{9}{x}$ where x is a positive number, and find the x that produces this number.

Let's tackle a more challenging problem that involves this inequality in a lesser-known way.

Example 3.2. Find the minimum value of $\frac{9x^2+36x+52}{12(x+2)}$ given that x is a positive integer.

Solution. This seems really daunting at first; we have a second degree polynomial on the numerator and a linear polynomial on the denominator, and the fraction cannot be simplified from the surface. However, we can use what we know about inequalities to generate something useful. In the last few problems, we had an expression and the reciprocal of that fraction to generate a useful AM-GM inequality. Lets try to do that here, given that the denominator has the expression $x + 2$ written in it. We can rewrite the numerator to include $x + 2$ by completing the square, and this results in $\frac{9(x+2)^2+16}{12(x+2)}$. We then can separate the fraction into $\frac{9(x+2)^2}{12(x+2)} + \frac{16}{12(x+2)}$, which simplifies to $\frac{3}{4}(x+2) + \frac{4}{3(x+2)}$. Now we can use AM-GM on these two variables! Using AM-GM and multiplying both sides by 2 results in $\frac{3}{4}(x+2) + \frac{4}{3(x+2)} \geq 2$ (the right side cancels out), which means the minimum value of the expression is $\boxed{2}$. This occurs when $\frac{3}{4}(x+2) = 1$, or $x = -$ △

The AM-GM inequality is very useful and can be used in many problems where it asks for the maximum value or the minimum value of an expression, even if it doesn't seem obvious at first. This also extends to HM (the harmonic mean) and QM (the quadratic mean), though applications of these inequalities are less common.

Checkpoint 3.2. Find the minimum value of

$$f(x) = x + \frac{1}{x} + \frac{1}{x + \frac{1}{x}}$$

for $x > 0$. *Source: AoPS*

4 The Cauchy-Schwarz Inequality

Next, we will discuss a very important inequality. It doesn't show up that frequently in competition math, but it can be converted into vector form to give an extremely powerful result that is used in a wide range of advanced mathematics. Here, we'll discuss a form of the inequality that doesn't involve vectors.

Theorem 4.1. *(Cauchy-Schwarz) For real numbers a_i and b_i, we have the following:*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Equivalently,

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$$

Theorem 4.2. *Equality is achieved in the Cauchy-Schwarz inequality if and only if there exists some real number t such that $a_i = tb_i$ for $i = 1, 2 \ldots n$. In other words, when we take each a_i and divide by corresponding b_i, we should get the same value each time.*

To get a feel for this powerful inequality, let's take a look at the following examples.

Example 4.1. Let x, y, and z be real numbers such that $x^2 + y^2 + z^2 = 1$. Find the maximum value of $3x + 4y + 12z$.

Solution. Because we are trying to maximize, we aim to have the expression $3x+4y+12z$ on the left-hand side of our inequality. So we want to find a_i and b_i such that

$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = 3x + 4y + 12z.$$

Since we have 3 terms in our expression, we can reasonably infer that $n = 3$. Therefore, we want $a_1b_1 + a_2b_2 + a_3b_3 = 3x + 4y + 12z$. Setting $a_1 = 3$, $a_2 = 4$, $a_3 = 12$ and $b_1 = x$, $b_2 = y$, $b_3 = z$ and plugging this into Cauchy-Schwarz, we have

$$(3x + 4y + 12z)^2 \leq (3^2 + 4^2 + 12^2)(x^2 + y^2 + z^2)$$

We know that $x^2 + y^2 + z^2 = 1$. Plugging this into the RHS, we have

$$(3x + 4y + 12z)^2 \leq 169$$

Noting that $\sqrt{169} = 13$, we get

$$-13 \leq 3x + 4y + 12z \leq 13$$

We are not quite done yet! Remember, to have equality in Cauchy-Schwarz, we must have that $\frac{3}{x} = \frac{4}{y} = \frac{12}{z}$. Writing y and z in terms of x and plugging into $x^2 + y^2 + z^2 = 1$, we see that we can indeed achieve equality. Hence, the maximum value of $3x + 4y + 12z$ is $\boxed{13}$. △

Example 4.2. Find the minimum value of the expression $(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$ over the reals.

Solution. This looks quite like the right-hand side of the Cauchy-Schwarz Inequality - we have a product of two expressions, each of which are a sum of squares. Specifically, let's set $(a_1, a_2, a_3) = (x, y, z)$ and $(b_1, b_2, b_3) = \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$. This results in

$$\left(x \cdot \frac{1}{x} + y \cdot \frac{1}{y} + z \cdot \frac{1}{z}\right)^2 \leq (x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

Note that the LHS can be simplified greatly! $x \cdot \frac{1}{x} = 1$, hence the LHS is simply $3^2 = 9$. Hence, we have

$$9 \leq (x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

Therefore, the expression will attain a value of 9 when we have $x = t \cdot \frac{1}{x}$, $y = t \cdot \frac{1}{y}$, and $z = t \cdot \frac{1}{z}$ for some t (from the equality case of Cauchy-Schwarz). Solving, we have $x = y = z = \sqrt{t}$ or $x = y = z = -\sqrt{t}$ for all real numbers t. This is equivalent to having $x = y = z$. Therefore, the expression $(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$ is minimized when $x = y = z$, giving the expression a value of $\boxed{9}$. △

Checkpoint 4.1. Show that if we have $a_i = tb_i$ for some real number t, then equality is achieved in the Cauchy-Schwarz inequality.

Checkpoint 4.2. Let a and b be real numbers. Find the maximum value of $a\cos\theta + b\sin\theta$ in terms of a and b. *Source: AoPS*

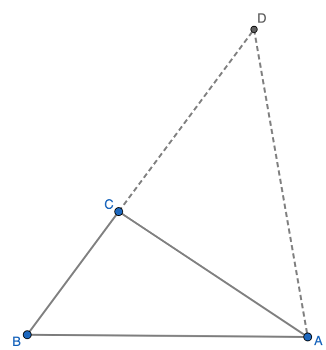
Checkpoint 4.3. Let a, b, and t be real numbers such that $a + b = t$. Find, in terms of t, the minimum value of $a^2 + b^2$. *Source: AoPS*

5 Geometric Inequalities

Inequalities don't have to just be algebraic. There are also geometric inequalities, the most important one being the triangle inequality, as we'll see in the following theorem. In this section, we will discuss the application of these inequalities in geometry problems. We will omit the discussion of vectors, but if you are familiar with them, we encourage you to try to work with and prove the inequalities in this section using vectors.

Theorem 5.1. *For any triangle, we have that any of the side lengths is less than the sum of the other two. In particular, for a triangle with side lengths a, b, and c with $a \leq b \leq c$, we have $c < b + a$.*

Proof. Let our triangle be ΔABC, with $BC = a$, $CA = b$, and $AB = c$. Suppose that $a \leq b \leq c$. Now extend $\overline{BC}$ to point D so that $CD = AD$, as shown in the following diagram.



From our construction, we have that ΔACD is isosceles, implying that $\angle CAD = \angle ADC$. Furthermore, we have that $\angle BAD > \angle CAD = \angle ADC = \angle ADB$. Now, let's consider ΔABD. Since the side opposite a larger angle in a triangle is larger (a consequence of the law of sines), we have that $BD > AB$. We can write $BD = BC + CD = BC + CA = a + b$. Since $AB = c$, we get $a + b > c$, as desired. □

Example 5.1. Find the sum of the smallest possible value and largest possible value of integer x if a triangle with sides 5, 7, and x exist.

Solution. For this problem, we will need to consider two cases. The first case is when x is greater than 7, making it the longest side. For this case, we get $5 + 7 > x$, which means x can be at most 11 (since it is integer). The second case is when x is less than 7, making 7 the longest side. This would mean $5 + x > 7$, or $x \geq 2$. This gives us that the smallest value of x is 3. Hence, the sum we desire is $11 + 3 = \boxed{14}$. △

We can also extend the triangle inequality to polygons.

Theorem 5.2. *For any polygon, any side length is less than the sum of the other sides. Furthermore, the non-negative real numbers $s_1, s_2 \ldots s_n$ can be the sides of a polygon if and only if for $i = 1, 2 \ldots n$, $s_i < \sum_{j=1, j \neq i}^{n} s_j$.*

Many of you may already be familiar with the Pythagorean theorem - however, it can be extended (as an inequality) to other triangles that are not right. Take a look at the theorem below:

Theorem 5.3. *Given a triangle ΔABC with side lengths a, b and c with $a \leq b \leq c$, we have the following:*

1. $a^2 + b^2 < c^2 \Leftrightarrow \Delta ABC$ *is obtuse.*

2. $a^2 + b^2 = c^2 \Leftrightarrow \Delta ABC$ *is right.*

3. $a^2 + b^2 > c^2 \Leftrightarrow \Delta ABC$ *is acute.*

Proof. This theorem can be proved by using the Law of Cosines and noting the sign of $\cos(C)$ in each of the three cases. □

Checkpoint 5.1. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod? *S*ource: AMC 10

6 Exercises

1. ⋆ In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

   (A) 43 (B) 44 (C) 45 (D) 46 (E) 47

   Source: AMC

2. ⋆ The lengths of the sides of a triangle with positive area are $\log_{10} 12$, $\log_{10} 75$, and $\log_{10} n$, where n is a positive integer. Find the number of possible values for n. *Source: AIME*

3. ⋆ A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly .500. During the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her win ratio is greater than .503. What's the largest number of matches she could've won before the weekend began? *Source: AIME*

4. ⋆ Which of the following quantities is the largest?

   $$A.\ \frac{2006}{2005}+\frac{2006}{2007} \qquad B.\ \frac{2006}{2007}+\frac{2008}{2007} \qquad C.\ \frac{2007}{2006}+\frac{2007}{2008}$$

   Source: Mandelbrot

5. ⋆ A rectangular box has volume 216. Find the smallest possible surface area of the box. *Source: AoPS*

6. ⋆⋆ Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 1$. Find the minimum value of

   $$a^2 + b^2 + c^2 + d^2.$$

   Source: AoPS

7. $\star\star$ Find the minimum value of
$$2\sqrt{x}+\frac{1}{x}$$
for $x>0$. *Source: AoPS*

8. $\star\star$ Let x be a positive real number. Find the minimum value of $4x^5+5x^{-4}$. *Source: AoPS*

9. $\star\star$ Show that $\sum_{k=1}^{n} a_k^2 \geq a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1$.

10. $\star\star$ Let A, M, and C be nonnegative integers such that $A+M+C=10$. What is the maximum value of $A\cdot M\cdot C+A\cdot M+M\cdot C+C\cdot A$? *Source: AMC*

11. $\star\star$ If $a\geq b>1$, what is the largest possible value of $\log_a(a/b)+\log_b(b/a)$? *Source: AMC*

12. $\star\star$ Let a,b, and c be real numbers such that
$$a+b+c=2, \text{ and}$$
$$a^2+b^2+c^2=12$$
What is the difference between the maximum and minimum possible values of c? *Source: AMC*

13. $\star\star$ Find all values of x satisfying $\frac{x-8}{x+5}+4\geq 3$. *Source: AoPS*

14. $\star\star$ Find all integer solutions x,y,z of the equation $x^2+5y^2+10z^2=4xy+6yz+2z-1$. *Source: AoPS*

15. $\star\star$ Find the least real number K such that for all real numbers x and y, we have $(1+20x^2)(1+19y^2)\geq Kxy$. *Source: Math Prize For Girls*

16. $\star\star$ Let a,b, and c be real numbers such that
$$a+b+c=2, \text{ and}$$
$$a^2+b^2+c^2=12$$
What is the difference between the maximum and minimum possible values of c? *Source: AMC*

17. $\star\star\star$ There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial x^3-ax^2+bx-a are real. In fact, for this value of a the value of b is unique. What is this value of b? *Source: AMC*

18. $\star\star\star$ Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32?

(A) 560 **(B)** 564 **(C)** 568 **(D)** 1498 **(E)** 2255
Source: AMC

19. $\star\star\star$ Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of

$$a^{(\log_{10} b)^2}$$

can be written in the form 10^x for some real number x. Find x. *Source: HMMT*

20. $\star\star\star$ For $1 \leq i \leq 215$ let $a_i = \dfrac{1}{2^i}$ and $a_{216} = \dfrac{1}{2^{215}}$. Let $x_1, x_2, ..., x_{216}$ be positive real numbers such that $\sum_{i=1}^{216} x_i = 1$ and $\sum_{1 \leq i < j \leq 216} x_i x_j = \dfrac{107}{215} + \sum_{i=1}^{216} \dfrac{a_i x_i^2}{2(1-a_i)}$. Find the maximum possible value of x_2. *Source: AIME*