Iowa City Math Circle Handouts
Algebra Techniques

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June 14, 2020

1 Solving Systems of Equations

Many math contest problems involve solving systems of linear equations. We will assume you know how to solve a basic system of 2 equations with 2 variables, as it is commonly taught in Algebra 1. Similar to how we solve a system of linear equations in 2 variables, we can also solve higher-dimensional systems using the same two methods: elimination and substitution. Recall that these methods can be summarized as follows.

Elimination:

1. Choose a variable and any equation containing that variable.
2. Solve for that variable in terms of the others by rearranging the equation.
3. Substitute this in for the chosen variable everywhere in the system.

Substitution:

1. Choose a variable and two equations containing it.
2. Add/subtract a multiple of one of these equations to the other equation in order to eliminate on of the variables from the result.
3. Repeat the above step for all pairs of equations containing the variable so that you end up eliminating the chosen variable from the system entirely.

We see that by applying both of these methods to a variable, we can eliminate the variable from the system, giving us an equivalent system of one less dimension. We repeatedly do this to get our solution to the system. It's not always guaranteed that our system has a solution, but our heuristic helps us determine if this is the case not only in linear systems, but also in some non-linear systems. We will see this in later examples.

First, let's take a look at the following linear system problem, which was inspired by a problem on the 2020 MATHCOUNTS State Sprint Round.

Example 1.1. If A, B, C, D, E, F and G satisfy the equations shown, what is the value of each variable?

$$\begin{aligned} A + B + C &= 5 \\ B + C + D &= 7 \\ C + D + E &= 9 \\ D + E + F &= 11 \\ E + F + G &= 13 \\ F + G &= 10 \\ A + F + G &= 11 \end{aligned}$$

Solution. We can take a look at the last two equations. When we subtract $F + G = 10$ from $A+F+G = 11$, we get $A = 1$. We can plug this value of A into the first equation to get $B+C = 4$. Since $B+C = 4$ and $B+C+D = 7$, we get $D = 3$. This means that $E + F = 8$ from the fourth equation. Subtracting this from $E + F + G = 13$ results in $G = 5$. After this, the rest of the problem is straightforward. The sixth equation means $F = 5$, and using the earlier equation $E + F = 8$ gives us $E = 3$. Now, we can use the third equation to yield $C = 3$, which means $B = 1$ with $B + C = 4$. To summarize, we have found that the values of the variables A through G, respectively, are 1, 1, 3, 3, 3, 5, and 5. △

Keep in mind, finding the values of all the variables in a system of equations may not be necessary (or even possible!) to find the value of an expression involving the variables. Let's consider the following example.

Example 1.2. Find the sum of of the 5 variables if the following equations are true:

$$\begin{aligned} A + B + C + D + 2E &= 10 \\ A + B + C + 2D + E &= 9 \\ A + B + 2C + D + E &= 8 \\ A + 2B + C + D + E &= 7 \\ 2A + B + C + D + E &= 8 \end{aligned}$$

Solution. There are 5 equations with 5 variables each, making solving for each variable difficult. However, we are only looking to find the sum of all of the variables. Notice that we can add all the equations to get $6A + 6B + 6C + 6D + 6E = 42$. Then, we can divide this equation by 6 to find that the sum of all variables is $\boxed{7}$. △

Notice that a quick way to find the sum of the variables in general for a linear system (as we did in the previous example) is to see if you can sum some subset of the equations in such a way that the coefficients of each of the variables in the sum is the same. This technique can also be extended to find the product of the variables of a system, as shown in the following example.

Example 1.3. Find all ordered triples (a, b, c) such that

$$\begin{aligned} ab &= 2 \\ bc &= 3 \\ ac &= 6. \end{aligned}$$

Solution. We can multiply every equation together to get $(abc)^2 = 36$ which simplifies to $abc = 6$ or -6. From there, we can plug in our given equations to our last equation to find the value of a, b, and c. When $abc = 6$, $a = 2$, $b = 1$, and $c = 3$. If $abc = -6$, then $a = -2$, $b = -1$, and $c = -3$. Therefore, our ordered triples are $(2, 1, 3)$ and $(-2, -1, -3)$. △

In the above solution, a common pitfall is to forget the $abc = -6$ case. When doing operations such as a square root, we need to make sure we include the negative root as well!

Checkpoint 1.1. Find $a + b$ (without explicitly finding a and b) if a and b satisfy $a + 2b = 20$ and $4a + 3b = 15$. After doing this, solve explicitly for a and b.

2 Special Factorizations

There are several special polynomial factorizations that are often useful in solving algebra problems. The most common ones are marked with a $\star$. Feel free to verify that these factorizations are correct by expanding.

1. $\star$ $x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right)$ In particular,
   (a) $x^2 - y^2 = (x + y)(x - y)$
   (b) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
   (c) $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + 1)$
2. $\star$ If n is odd,
   $$x^n + y^n = (x + y)\left(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}\right)$$
   In particular,
   (a) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
   (b) $x^n + 1 = (x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + (-1)^{n-1})$
3. $\star$ The following factorizations can help you find the powers of the sum of roots.

(a) $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

(b) $(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

For example, if $r_1, r_2, \ldots, r_k$ are the roots of a polynomial, we can find $r_1^n + r_2^n + \ldots + r_k^n$ by using Vieta's formulas and the factorizations above. This technique is known as Newton's Sums. Note that the above factorizations can be generalized, but we omit the general factorization since it doesn't show up that often competition math and it can be easily derived.

4. $\star$ (Simon's Favorite Factoring Trick, or SFFT for short) $xy+ax+by+ab = (x+b)(y+a)$. While this is simply foiling, it's important that you look out for expressions of this form.

5. Sophie Germain's: $a^4+4b^4 = (a^2-2ab+2b^2)(a^2+2ab+2b^2)$.

Proof.

$$\begin{aligned} a^4+4b^4 &= (a^4+4a^2b^2+4b^4)-(4a^2b^2) \\ &= (a^2+2b^2)^2-(2ab)^2 \\ &= (a^2+2b^2-2ab)(a^2+2b^2+2ab) \\ &= (a^2-2ab+2b^2)(a^2+2ab+2b^2) \end{aligned}$$

□

6. $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = \frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$.

   (a) $a^3+b^3+c^3 = 3abc$ if and only if $a+b+c=0$ or $a=b=c$.

   (b) If you have the terms a^3, b^3, and ab, then you can factorize it in this manner while allowing $c = \pm 1$.

7. (Lagrange) The abundance of squares formula:

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2+(ad-bc)^2.$$

Example 2.1. Find $3x^2y^2$ if x and y are integers such that $y^2+3x^2y^2 = 30x^2+517$. *Source: AIME*

Solution. We first rearrange the terms slightly to get

$$3x^2y^2+y^2-30x^2 = 517.$$

It looks like the LHS can be factored with SFFT; however, the constant term is missing. If we first try to factor it, we find that the constant term is 10:

$$(3x^2+1)(y^2-10) = 517-10.$$

So, we subtract 10 from both sides. Since x and y are integers, the two binomials must also be integers; thus, they must be factors of 507. $507 = 3 \cdot 13^2$. From inspection, $y^2-10 = 39$ and $3x^2+1 = 13$. This gives $y^2 = 49$ and $x^2 = 4$. Thus $3x^2y^2 = \boxed{588}$. △

Example 2.2. Compute the value of

$$\frac{2014^4 + 4 \cdot 2013^4}{2013^2 + 4027^2} - \frac{2012^4 + 4 \cdot 2013^4}{2013^2 + 4025^2}.$$

Source: BMO

Solution. First, we set $x = 2013$. We can rewrite the whole expression in terms of x:

$$\frac{(x+1)^4 + 4 \cdot x^4}{x^2 + (2x+1)^2} - \frac{(x-1)^4 + 4 \cdot x^4}{x^2 + (2x-1)^2}.$$

Now, we apply Sophie Germain's to the numerators of the expressions. We have

$$\begin{aligned}(x+1)^4 - 4x^4 &= ((x+1)^2 - 2(x+1)(x) + 2x^2) \cdot ((x+1)^2 + 2(x+1)(x) + 2x^2) \\ &= (x^2+1) \cdot (5x^2 + 4x + 1)\end{aligned}$$

and

$$\begin{aligned}(x-1)^4 - 4x^4 &= ((x-1)^2 - 2(x-1)(x) + 2x^2) \cdot ((x-1)^2 + 2(x-1)(x) + 2x^2) \\ &= (x^2+1) \cdot (5x^2 - 4x + 1).\end{aligned}$$

Expanding the denominators, we have $x^2 + (2x+1)^2 = 5x^2 + 4x + 1$ and $x^2 + (2x-1)^2 = 5x^2 - 4x + 1$. Therefore, our expression becomes

$$\begin{aligned}\frac{(x^2+1)(5x^2+4x+1)}{5x^2+4x+1} - \frac{(x^2+1)(5x^2-4x+1)}{5x^2-4x+1} \\ &= (x^2+1) - (x^2+1) \\ &= 0\end{aligned}$$

$\triangle$

Example 2.3. Let a, b, and c be real numbers such that $a + b + c = 1$. Show that

$$a^3 + b^3 + c^3 - 1 = 3(abc - ab - bc - ca).$$

Source: AwesomeMath

Solution. To prove this, like many other algebra problems, we first want to see which of the special factorizations listed above "look" like the statement we need to prove. The most obvious candidate is the factorization $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$. To incorporate this, we bring the $3abc$ term on the right-hand side of the desired equation to the other side and move the constant term to the other side to get

$$a^3 + b^3 + c^3 - 3abc = 1 - 3(ab + bc + ca).$$

Now, we can apply the factorization to the left-hand side to get

$$(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 1 - 3(ab + bc + ca).$$

Since $(a+b+c)=1$, this reduces to

$$a^2+b^2+c^2-ab-bc-ca=1-3(ab+bc+ca).$$

Bringing all of the non-constant terms on the right-hand side to the other side, we get

$$a^2+b^2+c^2+2(ab+bc+ca)=1.$$

Notice that the left-hand side is just the expansion of $(a+b+c)^2$, one of the factorizations on the list! Thus, it suffices to show

$$(a+b+c)^2=1.$$

This is trivial since we are given $a+b+c=1$. Hence we are done. $\triangle$

Checkpoint 2.1. Solve the system of equations

$$\begin{cases} x+y=xy-11 \\ y+z=yz-19 \\ z+x=zx-14 \end{cases}$$

3 Radicals and Conjugates

An expression that contains a *radical* contains a $\sqrt{\ }$ symbol. This could denote a square root, but it could also mean a cube root, $\sqrt[3]{\ }$, a fourth root, $\sqrt[4]{\ }$, or any other higher-power root.

The conjugate of a binomial expression $a+b$ is the binomial with an opposite sign for the second term: $a-b$. For a radical expression $a+\sqrt{b}$, the conjugate would be $a-\sqrt{b}$. We can denote the conjugate of an expression with a horizontal line over the expression: $\overline{a+\sqrt{b}}=a-\sqrt{b}$. Conjugates are useful because if we multiply a radical expression $a+\sqrt{b}$ by its conjugate $a-\sqrt{b}$ (difference of squares!), we eliminate the radical to get a^2-b. This is key to rationalizing the denominator or numerator in radical fractions, and can also be used to eliminate radicals in algebraic expressions.

Checkpoint 3.1. After rationalizing the numerator of $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}$, find the denominator in simplest form. *Source: AHSME*

3.1 Breaking up Nested Radicals

While solving a problem, you may obtain a radical expression containing a square root inside of a square root, such as $\sqrt{10+2\sqrt{21}}$. "De-nesting" the radical may be necessary to obtain the final answer. To do so, some algebra is needed.

Example 3.1. De-nest $\sqrt{10+2\sqrt{21}}$.

Solution. Let $\sqrt{a}+\sqrt{b}=\sqrt{10+2\sqrt{21}}$. Then

$$\begin{aligned}(\sqrt{a}+\sqrt{b})^2 &= 10+2\sqrt{21}\\ \implies a+2\sqrt{ab}+b &= 10+2\sqrt{21}\end{aligned}$$

Thus $a+b=10$ and $ab=21$. We could solve this system of equations, but in this case it's fairly clear that $a=3$ and $b=7$ (or vice versa). So $\sqrt{10+2\sqrt{21}}=\sqrt{3}+\sqrt{7}$. △

Checkpoint 3.2. De-nest $\sqrt{11+2\sqrt{30}}$.

All our calculations involving de-nesting can be generalized in the following theorem.

Theorem 3.1. *The following holds:*

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

Proof. This can be shown by squaring both sides, as shown below.

$$\begin{aligned}a+\sqrt{b} &= \frac{a+\sqrt{a^2-b}}{2}+\frac{a-\sqrt{a^2-b}}{2}+2\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\cdot\sqrt{\frac{a-\sqrt{a^2-b}}{2}}\\ &= \frac{a+\sqrt{a^2-b}+a-\sqrt{a^2-b}}{2}+2\sqrt{\frac{a+\sqrt{a^2-b}}{2}\cdot\frac{a-\sqrt{a^2-b}}{2}}\\ &= a+2\sqrt{\frac{a^2-(a^2-b^2)}{4}}\\ &= a+2\sqrt{\frac{b^2}{4}}\\ &= a+\sqrt{b}.\end{aligned}$$

□

From this theorem, we can de-nest radicals by simply plugging numbers into an expression (no equations involved!). As a result, this theorem is useful to memorize for math competitions.

4 Clever Substitutions

Substitution can be a powerful tool to simplify an algebraic expression. Unlike substituting to solve a system of equations, we can also introduce a new variable to reduce the complexity of an algebraic expression. In general, it is useful to substitute out complex expressions (such as radicals and square roots) or terms that appear often in an equation. The following two examples illustrate this technique.

Example 4.1. What is the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$? *Source: AIME*

Solution. Let $y = x^2+18x+30$. Substituting y into the equation gives us $y = 2\sqrt{y + 15}$. We can then square both sides and solve the resulting quadratic:

$$\begin{aligned} y^2 &= 4y + 60 \\ y^2 - 4y - 60 &= 0 \\ (y - 10)(y + 6) &= 0 \end{aligned}$$

This gives us the solution $y = 10$. $y = -6$ is an extraneous solution because the square root $\sqrt{y + 15}$ on the RHS cannot produce a negative value.

Plugging 10 back into our original substitution gives us

$$\begin{aligned} 10 &= x^2 + 18x + 30 \\ 0 &= x^2 + 18x + 20 \end{aligned}$$

We calculate the discriminant to check whether the roots are real: $18^2 - 4 \cdot 1 \cdot 20 = 244 > 0$. Thus, there are two real solutions to the equation. Using Vieta's, the product of the roots will be $\boxed{20}$. Alternatively, we can also find the product of the roots manually with the quadratic formula: the two roots will be

$$x = \frac{-18 + \sqrt{244}}{2}, x = \frac{-18 - \sqrt{244}}{2}$$

and their product will be $\boxed{20}$. △

Example 4.2. Find all real numbers a, b, and c such that

$$\sqrt[3]{a - b} + \sqrt[3]{b - c} + \sqrt[3]{c - a} = 0$$

Source: AwesomeMath

Solution. Let's replace the radicals with the variables x, y, and z, to get equation

$$x + y + z = 0.$$

Now, using one of the special factorizations mentioned in a previous section, we get

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x + y + z)\left(x^2 + y^2 + z^2 - xy - yz - zx\right) \\ &= 0 \cdot \left(x^2 + y^2 + z^2 - xy - yz - zx\right) \\ &= 0. \end{aligned}$$

Furthermore, observe that

$$x^3 + y^3 + z^3 = a - b + b - c + c - a = 0.$$

Thus, have that $0 - 3xyz = 0$, or $3xyz = 0$. This implies that at least one of x, y, and z must be 0. If $x = 0$, then $a = b$ and the original constraint of the problem is always satisfied. Extending this for the cases $y = 0$ and $z = 0$, we see that the answer is all ordered pairs (a, b, c) subject to either $a = b$, $b = c$, or $c = a$.

△

Checkpoint 4.1. Let $S = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 4(x-1) + 1$. Then S equals:

(A) $(x-2)^4$ **(B)** $(x-1)^4$ **(C)** x^4 **(D)** $(x+1)^4$ **(E)** x^4+1

Source: AHSME

5 Symmetry

Before we delve into this section, let's take a look at a basic problem.

Example 5.1. Given that $x + \frac{1}{x} = 5$, compute the value of $x^2 + \frac{1}{x^2}$.

Solution. We can try to solve for x directly, but that would get messy real quick! Let's try to look for a cleaner approach. We were given an equation with x and the expression we want to find contains x^2. So, lets try squaring the first equation! Doing so yields $x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} = 25$. Note that $2 \cdot x \cdot \frac{1}{x}$ is just 2, hence, we can obtain the desired expression by subtracting 2 from both sides. Doing so, we have $x^2 + \frac{1}{x^2} = \boxed{23}$. △

This is just one example of a problem involving the expression $x + \frac{1}{x}$. Squaring the expression is a common way to deal with this expression. Below is a harder problem involving $x + \frac{1}{x}$ (although at first glance, it may not be obvious that $x + \frac{1}{x}$ is involved!)

Example 5.2. Find all the solutions to the following equation: $x^4 + 3x^3 - 8x^2 + 3x + 1 = 0$.

Solution. This equation is a fourth degree polynomial, which are usually not easy to handle. However, we can exploit one of the properties of this particular polynomial: the coefficients are symmetric! What do we mean by this? A polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is symmetric if $a_{n-k} = a_k$ for $k = 0, 1 \ldots n$. In other words, if you were to list out the coefficients of the polynomial, it would read the same backwards and forwards. This symmetry can be easily connected to the expression $x + \frac{1}{x}$, but we first need a way to generate fractions. To do this, we can divide the polynomial by x^2, which gives us the equation $x^2 + 3x - 8 + \frac{3}{x} + \frac{1}{x^2} = 0$. Now, the key step in this problem is to define a new variable y such that $y = x + \frac{1}{x}$. To deal with the x^2 and its reciprocal, we square both sides of $y = x + \frac{1}{x}$ (as we did in the previous problem) to obtain $x^2 + \frac{1}{x^2} = y^2 - 2$ (after subtracting 2 from both sides). Now, we are ready to write our expression in x in terms of y. We have

$$\begin{aligned} x^2 + 3x - 8 + \frac{3}{x} + \frac{1}{x^2} &= \left(x^2 + \frac{1}{x^2}\right) + 3x + \frac{3}{x} - 8 \\ &= (y^2 - 2) + 3\left(x + \frac{1}{x}\right) - 8 \\ &= y^2 + 3y - 10. \end{aligned}$$

Therefore, we have $y^2 + 3y - 10 = 0$, which yields two solutions for y: -5 and 2. Now, we must solve for x. For $y = 2$, we have $x + \frac{1}{x} = 2$. Multiplying both sides by x, we have $x^2 + 1 = 2x$, or $x^2 - 2x + 1 = 0$, a quadratic in x. Factoring the quadratic, we have $(x-1)^2 = 0$, so $x = 1$. For $y = -5$, we have $x + \frac{1}{x} = -5$, resulting in the quadratic $x^2 + 5x + 1$. Using the quadratic formula, we get the solutions $x = \frac{-5+\sqrt{21}}{2}$ and $x = \frac{-5-\sqrt{21}}{2}$. Putting this all together, our solutions to the original equation are $x = \frac{-5+\sqrt{21}}{2}, \frac{-5-\sqrt{21}}{2}$, and 1. △

In general, we can use this technique to find the roots of any symmetric polynomial. To summarize, we take the following steps:

1. Divide the polynomial by the middle term (ignoring the coefficient). For example, divide the polynomial $x^5 + 4x^2 - 3x^3 + 4x + 1$ by x^3, and the polynomial $x^6 - 3x^5 + x^3 + 3x + 1$ by x^3. More generally, if your polynomial has degree n, divide by $x^{\lceil \frac{n}{2} \rceil}$.
2. Let $y = x + \frac{1}{x}$, and write the expression in terms of y by writing $x + \frac{1}{x}$, $x^2 + \frac{1}{x^2} \dots x^{\lceil \frac{n}{2} \rceil} + \frac{1}{x^{\lceil \frac{n}{2} \rceil}}$ in terms of y.
3. You should now have a polynomial in y. Find the roots of this polynomial.
4. Now, solve for x by setting $x + \frac{1}{x}$ to all the solutions you got for y. For each value of y, multiply both sides by x to get a quadratic in x. From here, you can either factor the expression or use the quadratic formula to solve.

Below are some additional possible forms of $y = x + \frac{1}{x}$ that are convenient to use:

$$x^2 + \frac{1}{x^2} = y^2 - 2$$
$$x^3 + \frac{1}{x^3} = y^3 - 3y$$
$$x^4 + \frac{1}{x^4} = y^4 - 4y^2 + 2$$

Overall, symmetry can be a useful tool when dealing with specific polynomials. It can also be employed when the constraints in the problem are symmetric with respect to the variables, as in the following example.

Example 5.3. In the following system of equations, solve for the product xyz.

$$x + \frac{1}{y} = 4$$
$$y + \frac{1}{z} = 1$$
$$z + \frac{1}{x} = \frac{7}{3}$$

Source: AMC

Solution. To get the term xyz, we can try to multiply the three equations, giving us

$$\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)=xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}$$

by simple foiling. By simple rearrangement and substitution, we have

$$\begin{aligned}\frac{28}{3}&=\left(x+\frac{1}{y}\right)\left(y+\frac{1}{z}\right)\left(z+\frac{1}{x}\right)\\&=xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}\\&=xyz+\frac{1}{xyz}+\left(x+\frac{1}{y}\right)+\left(y+\frac{1}{z}\right)+\left(z+\frac{1}{x}\right)\\&=xyz+\frac{1}{xyz}+4+1+\frac{7}{3}\\&=xyz+\frac{1}{xyz}+\frac{22}{3}.\end{aligned}$$

Thus,

$$xyz+\frac{1}{xyz}=2.$$

Letting $w=xyz$, we get the following quadratic in w:

$$w^2-2w+1=0.$$

This should be recognizable as $(w-1)^2$, so the only solution is $w=xyz=\boxed{1}$. △

Checkpoint 5.1. In the following system of equations, solve for the quantity $xyz+\frac{1}{xyz}$ in terms of a, b, and c.

$$\begin{aligned}x+\frac{1}{y}&=a\\y+\frac{1}{z}&=b\\z+\frac{1}{x}&=c\end{aligned}$$

6 Exercises

1. ⋆ Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

   (A) 22 (B) 60 (C) 119 (D) 194 (E) 231

   Source: AMC

2. $\star$ For real numbers w and z,

$$\frac{\frac{1}{w}+\frac{1}{z}}{\frac{1}{w}-\frac{1}{z}}=2014.$$

What is $\frac{w+z}{w-z}$? *Source: AMC*

3. $\star$ Suppose that real number x satisfies

$$\sqrt{49-x^2}-\sqrt{25-x^2}=3$$

What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$? *Source: AMC*

4. $\star$ The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers? *Source: AMC*

5. $\star\star$ Real numbers x and y satisfy $x+y=4$ and $x\cdot y=-2$. What is the value of

$$x+\frac{x^3}{y^2}+\frac{y^3}{x^2}+y?$$

Source: AMC

6. $\star\star$ How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020}+y^2=2y?$$

Source: AMC

7. $\star\star$ All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products? *Source: AMC*

8. $\star\star$ If x is a real number such that $(x-3)(x-1)(x+1)(x+3)+16=116^2$, what is the largest possible value of x? *Source: Math Prize For Girls*

9. $\star\star$ Consider the following system of 7 linear equations with 7 unknowns:

$$\begin{aligned}
a+b+c+d+e&=1\\
b+c+d+e+f&=2\\
c+d+e+f+g&=3\\
d+e+f+g+a&=4\\
e+f+g+a+b&=5\\
f+g+a+b+c&=6\\
g+a+b+c+d&=7.
\end{aligned}$$

What is g? *Source: Math Prize For Girls*

10. $\star\star$ Find all real solutions to $x^4 + (2-x)^4 = 34$. *Source: HMMT*

11. $\star\star$ Let a, b, c, and x be reals with $(a+b)(b+c)(c+a) \neq 0$ such that

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \qquad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \qquad \textit{and} \qquad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

Compute x. *Source: HMMT*

12. $\star\star$ Suppose a, b, and c are real numbers such that

$$\frac{ac}{a+b} + \frac{ba}{b+c} + \frac{cb}{c+a} = -9$$

and

$$\frac{bc}{a+b} + \frac{ca}{b+c} + \frac{ab}{c+a} = 10.$$

Compute the value of

$$\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}.$$

Source: AoPS

13. $\star\star$ Find the sum of all real x such that $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$. *Source: AHSME*

14. $\star\star$ Given that $x + \sqrt{x^2 - 1} + \frac{1}{x - \sqrt{x^2-1}} = 20$, compute $x^2 + \sqrt{x^4 - 1} + \frac{1}{x^2 + \sqrt{x^4 - 1}}$ *Source: AHSME*

15. $\star\star$ Find $x^2 + y^2$ if x and y are positive integers such that $xy + x + y = 71$ and $x^2y + xy^2 = 880$. *Source: AIME*

16. $\star\star$ Find a_{17} if

$$\begin{aligned} a_1 + a_2 + a_3 &= 1, \\ a_2 + a_3 + a_4 &= 2, \\ &\vdots \\ a_{16} + a_{17} + a_1 &= 16, \\ a_{17} + a_1 + a_2 &= 17. \end{aligned}$$

Source: Mandelbrot

17. $\star\star$ Solve the following system of equations:

$$\begin{aligned} x(x+y+z) &= 24, \\ y(x+y+z) &= 18, \\ z(x+y+z) &= -6. \end{aligned}$$

18. $\star\star\star$ Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Find $\sin^{12} x + \cos^{12} x$. *Source: AIME*

19. $\star\star\star$ Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. Find the greatest possible value of $a^3 + b^3 + c^3$.
Source: AIME

20. $\star\star\star$ Compute

$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}.$$

Source: AIME