Chapter 14

Triangle Centers and Cevians

14.1 Definitions

Definition. The **incircle** of a triangle ΔABC is the circle which is tangent to the sides of the triangle (AB, BC, and CA). The incircle's radius is referred to as the **inradius**; its center is referred to as the **incenter**.

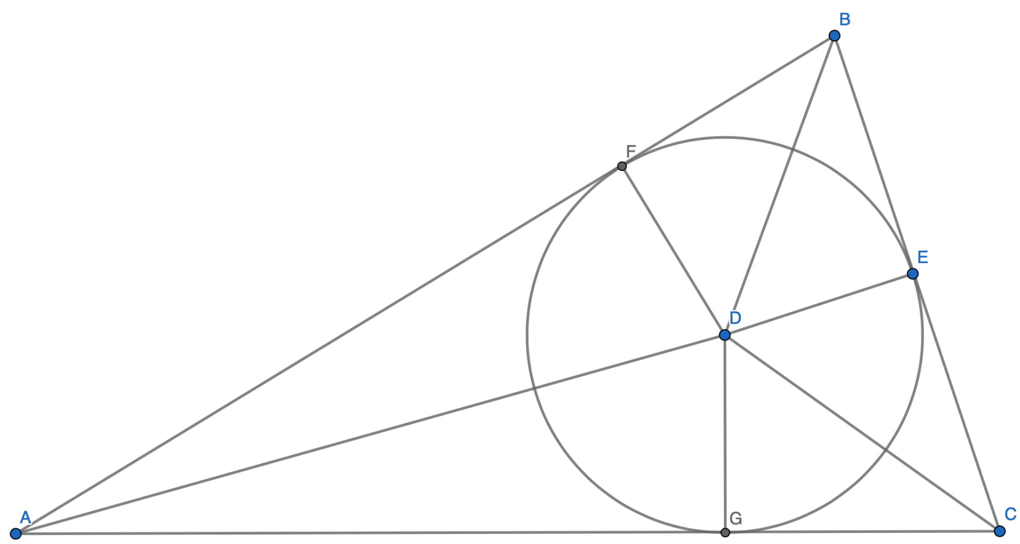
In general, the incircle of a polygon is the *unique* circle that is centered inside the polygon and is tangential to all the sides of the polygon. However, only triangles are guaranteed to have an incircle - other polygons may not have one.

Definition. An **angle bisector** of an angle $\angle BAC$ is the line through A such that for any point D on the line, $\angle BAD = \angle DAC$. When we say "angle bisectors" (in the context of a triangle), we are referring to the angle bisectors through the vertices of the triangle, with the angles being the ones formed by the sides of triangle.

As we will see in Theorem 10.1, angle bisectors and incircles relate very closely.

Theorem 14.1. *The angle bisectors of a triangle intersect at the incenter. Thus, we can redefine the incenter of a triangle as the intersection of its angle bisectors.*

Proof. We construct the following diagram.



We have that D is the incenter of ΔABC, and the incircle is tangent to the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ at points F, E, and G, respectively. It suffices to show that $\angle FBD = \angle EBD$, $\angle FAD = \angle GAD$, and $\angle GCD = \angle ECD$. We see that ΔFBD is congruent to ΔEBD since they are both right triangles, $FD = ED$, and they both share $\overline{BD}$. Thus, $\angle FBD = \angle EBD$. Similarly we get ΔFAD is congruent of ΔGAD (implying that $\angle FAD = \angle GAD$) and ΔGCD is congruent to ΔECD (implying that $\angle GCD = \angle ECD$). Hence we are done. $\square$

Definition. A **median** of a triangle is the line passing through one of its vertices and bisects the opposite side. For example, in triangle ΔABC, let D be the midpoint of side BC. Then the median from A is the line through AD.

This leads us to the following definition:

Definition. The **centroid** of a triangle is the point of concurrency of the medians of a triangle, implying that the medians of a triangle intersect at a single point. A property regarding centroids and medians is that in triangle ABC, with M as the centroid, and D as the intersection of line AM and segment BC, the ratio of $AM : MD$ is $2 : 1$. This works for all triangles and cevians (described later).

Definition. The **medial triangle** of a triangle ΔABC is the triangle ΔDEF where D is the midpoint of BC, E is the midpoint of BC, and F is the midpoint of CA. In other words, the medial triangle connects the midpoints of the sides of the original triangle.

Checkpoint 14.1. Show that the medial triangle divides any triangle into four congruent triangles.

Definition. The **orthocenter** of a triangle ΔABC is the intersection of its altitudes, implying that the altitudes of a triangle are concurrent.

Definition. A **cevian** of a triangle is a line that passes through a vertex of the triangle and the opposite side to that vertex. For example, in triangle ΔABC, let D be any point on BC. Then AD is a cevian of the triangle.

Angle bisectors, medians, and altitudes are examples of cevians of a triangle, and are therefore subjectable to Ceva's theorem (as we will see later on).

Definition. The **circumcircle** of triangle ΔABC is the circle that passes through points A, B, and C. The circumcircle's radius is considered to be the circumradius, while the center of the circumcircle is referred to as the circumcenter.

Triangles are guaranteed to have a circumcircle - other polygons may not have one. However, if a quadrilateral or any other non-triangle polygon have a circumcenter, that polygon is considered to be cyclic.

Theorem 14.2. *The perpendicular bisectors of a triangle intersect at the circumcenter. Thus, we can redefine the circumcenter of a triangle as the intersection of its perpendicular bisectors.*

Proof. We will split the proof into two parts: first, we will show that the perpendicular bisectors are concurrent (equivalently, that they all intersect at a single point). Lastly, we will show that this intersection point is equidistant from the vertices of the triangle.

To show that the perpendicular bisectors are concurrent, we will draw the medial triangle of ΔABC. Let this triangle be ΔDEF, as defined in the definition above. Then the perpendicular bisectors of ΔABC are the altitudes of ΔDEF. So we must show that the altitudes of ΔDEF are concurrent (in other words, we must show the existence of the orthocenter). We will come back to this in a later exercise using the angular form of Ceva's Theorem.

Now, to complete the second part of the proof, recall the properties of a perpendicular bisector. The perpendicular bisector of a segment AB is the locust of points that are equidistant from A and B. Let O be the point of intersection of the three perpendicular bisectors. Since O lies on these bisectors, we have that O is equidistant from A and B, O is equidistant from B and C, and O is equidistant from C and A. Combining, we have that O is equidistant from A, B, and C. □

14.2 Theorems

Theorem 14.3. *The area of any triangle is given by $A = rs$, where r is the inradius of the triangle and s is the semiperimeter (half of the perimeter) of the triangle.*

Proof. We use the same diagram as the one in the proof of *Theorem 10.1.* Let r be the radius of the incircle.

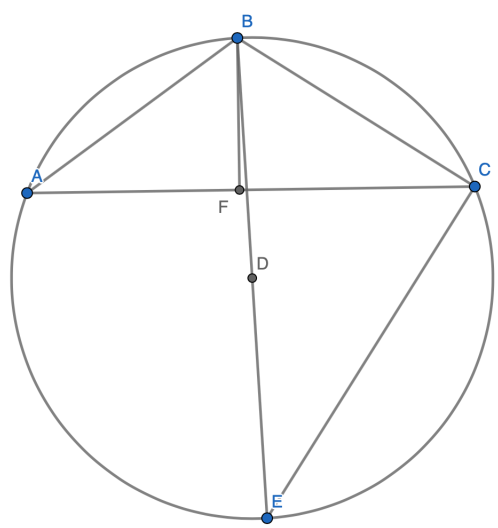
The area of ΔACD is given by $\frac{AC \cdot GD}{2} = \frac{AC \cdot r}{2}$ using the base $\overline{AC}$. Similarly, the area of ΔBCD is $\frac{BC \cdot ED}{2} = \frac{BC \cdot r}{2}$ and the area of ΔABD is $\frac{AB \cdot FD}{2} = \frac{AB \cdot r}{2}$. Hence, the area of triangle ΔABC is

$$\begin{aligned}\Delta ABC &= [\Delta ACD] + [\Delta BCD] + [\Delta ABD] \\ &= \frac{AC \cdot r}{2} + \frac{BC \cdot r}{2} + \frac{AB \cdot r}{2} \\ &= r \cdot \frac{AB + BC + CA}{2} \\ &= rs.\end{aligned}$$

□

Theorem 14.4. *For any triangle ΔABC, the area of the triangle can also be written as $A = \frac{abc}{4R}$, where A is the area of the triangle, R is the circumradius of the triangle, and a, b, c are the length of the sides of the triangle.*

Proof. We construct the following diagram.



In the above diagram, F is the foot of altitude $\overline{BF}$, D is the circumcenter of ΔABC, and E is a point on the circumcircle such that $\overline{BE}$ is a diameter. Let $a = AB$, $b = BC$, $c = CA$, A be the area of ΔABC, and R be the circumradius. Since any triangle inscribed in a semicircle is a right triangle, we have that $\angle BCE = 90°$. Additionally, we have that $\angle BAF = \angle BEC$ since they are both subtended by arc BC. Thus, by AAA similarity, ΔBAF is similar to ΔBEC. As a result, we have $\frac{BC}{BE} = \frac{BF}{BA}$ or $\frac{b}{2R} = \frac{BF}{a}$. Since $A = \frac{AC \cdot BF}{2} = \frac{c \cdot BF}{2}$, we have $BF = \frac{2A}{c}$. Substituting this in, we have $\frac{b}{2R} = \frac{2A}{ac}$ or $A = \frac{abc}{4R}$. □

Theorem 14.5. *(Angle Bisector Theorem) Let D be a point on BC in triangle ΔABC. Then AD is the angle bisector through A iff*

$$\frac{AB}{BD} = \frac{AC}{CD}$$

Checkpoint 14.2. Prove the Angle Bisector Theorem using similarity. (Hint: Draw the line through point C parallel to AB and then extend $\overline{AD}$ to this auxiliary line).

The following theorems don't show up that often in competition math but are still useful to know.

Theorem 14.6. *(Ceva's Theorem) Let AD, BE, and CF be cevians of a triangle ΔABC. Then AD, BE, and CF are concurrent iff*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Checkpoint 14.3. Prove Ceva's Theorem using areas.

Checkpoint 14.4. Prove the existence of the centroid (or that the medians of a triangle are concurrent) using Ceva's Theorem.

Theorem 14.7. *(Angular form of Ceva's Theorem) Let AD, BE, and CF be cevians of a triangle ΔABC. Then AD, BE, and CF are concurrent iff*

$$\frac{\sin(\angle DAB) \cdot \sin(\angle EBC) \cdot \sin(\angle FCA)}{\sin(\angle DAC) \cdot \sin(\angle EBA) \cdot \sin(\angle FCB)} = 1$$

Checkpoint 14.5. Prove the existence of the orthocenter (or that the altitudes of a triangle are concurrent) using the Angular form of Ceva's Theorem.

14.3 Exercises

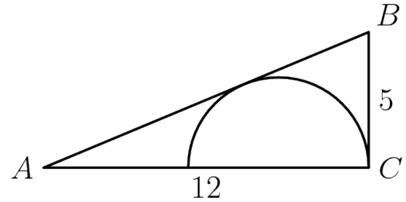
1. Find the radius of a circle which is inscribed in a triangle whose perimeter is 40 and area is 120. *Source: AoPS Community*

2. Let ABC be a triangle with side lengths 3, 4, and 5. What is the radius of the circumcircle of triangle ABC and the radius of the incircle of triangle ABC?

3. Find the circumradius of a triangle with sides 13, 14 and 15.

4. A line bisecting the larger acute angle in a triangle with sides of length 33, 44 and 55 cm divides the opposite side into two segments. What is the length of the shorter segment of that side? *Source: MATHCOUNTS*

5. In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



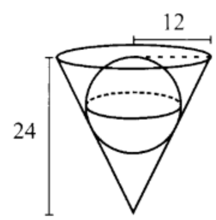
Source: AMC 8

6. Let DEF be a triangle with side lengths 8, 15, and 17. Let A be the incenter of DEF, and let B be the circumcenter of DEF. Find the length of segment AB.

7. Find the maximum inradius of a quadrilateral with side lengths of all 5. (Also figure out why that number is the maximum!)

8. What is the circumradius minus the inradius of a regular hexagon with side length 3?

9. A sphere is inscribed in a right cone with base radius 12 cm and height 24 cm, as shown. Find the radius of the sphere.



Source: MATHCOUNTS

10. A sphere with center A has radius 6. A triangle with sides of length $15, 15$, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between A and the plane determined by the triangle? *Source: AMC 10/12*