Chapter 2

Polynomials

2.1 Warm-up Problems

1. Find y: $\sqrt{19+3y}=7$. *Source: MATHCOUNTS*

2. How many terms are in the expansion of $(a+b+c)(d+e+f+g)$? *Source: Alcumus*

3. Find the solution(s), if any exist, (x, y) to each of the following systems of linear equations.

   (a) $\begin{cases} x+2y=3 \\ 2x+5y=8 \end{cases}$

   (b) $\begin{cases} x+2y=3 \\ 3x+6y=7 \end{cases}$

   (c) $\begin{cases} x+2y=3 \\ 3x+6y=9 \end{cases}$

2.2 What is a Polynomial?

A polynomial is a function of several variables consisting of different terms. Each term is expressed as a product of real multiples of non-negative integer powers of the variables. For this handout, we'll only be considering polynomials with a single variable and real coefficients.

A polynomial with only one term is called a **monomial** and a polynomial with two terms is called a **binomial**.

A general, single-variable polynomial can be written as $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. We refer to the set $\{a_0, a_1, \ldots, a_n\}$ as the coefficients of the polynomial. In addition, we call n the **degree** of the polynomial. A polynomial of degree 2 is called a **quadratic** polynomial and a polynomial of degree 3 is called a **cubic** polynomial (these are the polynomials we most frequently deal with).

2.3 Polynomial Arithmetic

2.3.1 Addition and Multiplication of Polynomials

To add two polynomials, we add the coefficients of like terms (terms that contain the same power of x). We can subtract two polynomials in a similar fashion. To multiply two polynomials, we use the FOILing technique (or in other words, the distributive property). In essence, we multiply each term in one polynomial with each term in the other polynomial, and add together all of these results. We illustrate these computations in the following example.

Example 2.1. Let $p(x) = 2x + 1$ and $q(x) = x^3 + 3x + 4$ be two polynomials. Find $p(x) + q(x)$ and $p(x) \cdot q(x)$.

Solution. We demonstrate the processes for addition and multiplication of polynomials discussed above. We have that

$$\begin{aligned} p(x) + q(x) &= (2x + 1) + \left(x^3 + 3x + 4\right) \\ &= x^3 + (2 + 3)x + (1 + 4) \\ &= \boxed{x^3 + 5x + 5} \end{aligned}$$

and

$$\begin{aligned} p(x) \cdot q(x) &= (2x + 1) \cdot \left(x^3 + 3x + 4\right) \\ &= (2x) \cdot x^3 + (2x) \cdot (3x) + (2x) \cdot 4 + 1 \cdot x^3 + 1 \cdot (3x) + 1 \cdot 4 \\ &= 2x^4 + 6x^2 + 8x + x^3 + 3x + 4 \\ &= \boxed{2x^4 + x^3 + 6x^2 + 11x + 4} \end{aligned}$$

△

2.3.2 Long Division with Polynomials

We can also divide polynomials using long division. This works in pretty much the same way as long division with integers.

Example 2.2. Find the quotient and remainder when $2x^4 + 4x^2 - 1$ is divided by $x + 1$.

Solution. We can use long division to divide the two polynomials:

$$\begin{array}{r|l} & 2x^3 - 2x^2 + 6x - 6 \\ \hline x + 1 & 2x^4 \qquad\quad + 4x^2 \qquad - 1 \\ & -\,2x^4 - 2x^3 \\ \hline & \qquad -\,2x^3 + 4x^2 \\ & \qquad 2x^3 + 2x^2 \\ \hline & \qquad\qquad 6x^2 \\ & \qquad\qquad -\,6x^2 - 6x \\ \hline & \qquad\qquad\qquad -\,6x - 1 \\ & \qquad\qquad\qquad 6x + 6 \\ \hline & \qquad\qquad\qquad\qquad 5 \end{array}$$

Hence, our quotient is $\boxed{2x^3 - 2x^2 + 6x - 6}$ and our remainder is $\boxed{5}$. △

2.3.3 Review Exercises

1. Find the polynomial $p(x)$ if

$$(x^2 - 3x + 5)p(x) = x^4 - 3x^3 + 15x - 25.$$

   Source: Alcumus

2. Find the quotient and remainder when $x^6 - 3$ is divided by $x + 1$. *Source: Alcumus*

2.4 Roots

Suppose $p(x)$ is a polynomial. The **roots** of $p(x)$ are the values of x that make $p(x) = 0$. This is essentially the same as solving the equation $p(x) = 0$. There are a few methods we can use to find the roots of a polynomial.

2.4.1 Factoring

We can factor a polynomial $p(x)$ of degree n and leading coefficient a into a product of n binomials, $a(x - r_1)(x - r_2)\cdots(x - r_n)$, where $r_1, r_2, \cdots, r_n$ are the roots of $p(x)$.

Every polynomial can be factored, however not every polynomial has a nice factorization in integers, or even real numbers. We'll be mainly focusing on factoring quadratic equations in this handout. For quadratics, it should usually be apparent if a quadratic can be factored nicely by looking at factors of the constant term. If not, you can find the roots using a different method like the quadratic formula or completing the square (which we'll discuss below). Moreover, to check if a general polynomial with integer coefficients has any rational roots, we use the following theorem.

Theorem 2.1. *(Rational Root Theorem) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_0, a_1, \ldots, a_n$ are integers and $a_0 \neq 0$. Then if $\frac{p}{q}$ is a rational root of $f(x)$ in lowest terms, then p divides a_0 and q divides a_n.*

Proof. Plugging in $\frac{p}{q}$ into $f(x)$ and multiplying by q^n to get integer coefficients, we have

$$q^n \cdot f\left(\frac{p}{q}\right) = 0 = a_n p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots + a_0 q^n$$

Notice that all of these terms besides $a_0 q^n$ are divisible by p, and since the sum of all the terms is divisible by p, we have that p divides $a_0 q^n$. Since p and q are relatively prime, p must divide a_0. Similarly, all of the terms besides $a_n p^n$ are divisible by q, and since the sum of all the terms is divisible by q, we have that q divides $a_n p^n$. Since p and q are relatively prime, q must divide a_n. Hence we are done. □

2.4.2 Vieta's Formulas

Sometimes a problem may involve the sums or products of the roots of a polynomial. In that case, there is a way to find the sums or products of the roots without needing to factor the polynomial. The formulas below are known as *Vieta's formulas*.

Suppose polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has a factorization $a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. Note that the product of the roots, $r_1 r_2 \cdots r_n$ is equal to $(-1)^n \frac{a_0}{a_n}$ (when we multiply all the binomials together, the product of all the roots and a_n will form the constant term).

Similarly, the sum of the roots $r_1 + r_2 + \cdots + r_n$ will equal $-\frac{a_{n-1}}{a_n}$ because while expanding the factorization, the x^{n-1} term will equal $-a_n(r_1 + r_2 + \cdots + r_n)x^{n-1}$.

In addition, the pairwise sum of the roots $r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n$ will equal $\frac{a_{n-2}}{a_n}$, the 3-wise sum of the roots $r_1 r_2 r_3 + \cdots + r_{n-2} r_{n-1} r_n$ will equal $-\frac{a_{n-3}}{a_n}$, and so on.

It can be helpful to remember for a quadratic equation $ax^2 + bx + c$ that the sum of the roots will be $-\frac{b}{a}$ and the product of the roots will be $\frac{c}{a}$ (feel free to prove this on your own).

Example 2.3. Find the sum of the values of x which satisfy $x^2 + 2019x = 1337$.

Solution. Eww, big numbers and a quadratic equation that may or may not be able to be factored. But it actually doesn't matter. With Vieta's, the sum of the roots of the equation is just $-b/a = -2019/1 = \boxed{-2019}$. △

Checkpoint 2.1. What is the sum of the roots of $x^2 - 5x + \pi \cos \frac{2\pi}{5}$?

Checkpoint 2.2. Find the product of the roots of $x^5 - 6x^4 + 1x^3 + 8x^2 - 9x + 10$.

2.4.3 Review Exercises

1. Find the largest value of c such that $\frac{c^2+6c-27}{c-3} + 2c = 23$. *Source: Alcumus*

2. Find the sum of all solutions to this equation: $x^2 + 6^2 = 10^2$. *Source: MATHCOUNTS*

3. The graphs of $y = x^2 - 7x + 7$ and the line $y = -2$ intersect at two points. What is the sum of the x-coordinates of these two points?

4. What is the sum of the squares of the solutions of $x^2 + 13x + 6 = 0$?

5. Find the sum of the squares of the roots of $x^3 + 5x^2 + 6x + 9$, assuming the roots are distinct.

2.5 Quadratic Polynomials

Quadratic polynomials are a special type of polynomial that show up in math competitions frequently. These polynomials are of the form $ax^2 + bx + c$ (where $a \neq 0$; in other words, they are single-variable and of degree 2. They show up frequently in competition math due to their relatively transparent nature.

2.5.1 Completing the Square

First, we will find the roots of the general quadratic $ax^2 + bx + c$ by completing the square, a technique that will enable us to find the roots of any quadratic. We have

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right] \end{aligned}$$

This is our completion of squares form. In this form, we can compute the roots of the polynomial easily by moving the constant term to one side and taking the square-root, to end up with a simple linear equation. We see that

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right] &= 0 \\ \implies \qquad a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \end{aligned}$$

Multiplying both sides by $4a$, we have

$$4a^2\left(x + \frac{b}{2a}\right)^2 = b^2 - 4ac$$

Since $a \neq 0$, dividing both sides by $4a^2$ and taking square roots, we get

$$\pm\left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a}$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, we have an explicit formula (known as the Quadratic Formula) for the roots of any quadratic. As seen above, completing the square helps us find the roots of any polynomials.

In general, to complete the square of any algebraic expression, you rewrite the expression as a sum of squares of polynomials and a constant. Coupling the technique of completion of squares with the idea that the sum of squares of real numbers is always non-negative is useful to solve algebraic equations (as seen in the following examples and exercises). Let's do an example to cement this technique.

Example 2.4. Complete the square of the quadratic $2x^2 + 3x + 4$.

Solution. We rewrite the original quadratic as

$$\begin{aligned} 2x^2+3x+4 &= 2\left(x^2+\frac{3}{2}x\right)+4 \\ &= 2\left(x+\frac{3}{4}\right)^2-2\cdot\left(\frac{3}{4}\right)^2+4 \\ &= \boxed{2\left(x+\frac{3}{4}\right)^2+\frac{23}{8}}. \end{aligned}$$

△

Example 2.5. Solve the following system of equation in real numbers (x, y, z) *Source: 105 Algebra Problems*

$$\begin{cases} x-y^2-z &= \frac{1}{3} \\ y-z^2-x &= \frac{1}{6} \end{cases}$$

Solution. To cancel out the x term, we add the two equations to get $y-y^2-z-z^2=\frac{1}{2}$. Multiplying both sides by -1, we have $y^2-y+z^2+z=-\frac{1}{2}$. Now, we can complete the square of y^2-y and z^2+z to get

$$\left(y-\frac{1}{2}\right)^2-\frac{1}{4}+\left(z+\frac{1}{2}\right)^2-\frac{1}{4}=-\frac{1}{2}$$

Cancelling the constant terms, we have that

$$\left(y-\frac{1}{2}\right)^2+\left(z+\frac{1}{2}\right)^2=0$$

Since the square of any real number is non-negative, we must have that $y-\frac{1}{2}=0$ and $z+\frac{1}{2}=0$. Hence, $y=\frac{1}{2}$ and $z=-\frac{1}{2}$. By plugging these values of y and z into the original equation $x-y^2-z=\frac{1}{3}$ and solving for x, we have that $x=\frac{1}{12}$. Thus, the only solution to this system is $x=\frac{1}{12}$, $y=\frac{1}{2}$, and $z=-\frac{1}{2}$. △

Checkpoint 2.3. Complete the square of the quadratic $3x^2-2x-1$.

2.5.2 The Discriminant

We define the discriminant of the quadratic $p(x)=ax^2+bx+c$ to be the expression b^2-4ac. Notice that this expression appears under a square root in the quadratic formula. As a result, we will analyze the implications of the sign of the discriminant. The following results are all direct consequences of the quadratic formula.

- If the discriminant is zero ($b^2-4ac=0$), then the quadratic has exactly one root (which is real).
- If the discriminant is positive ($b^2-4ac>0$), then the quadratic has two, distinct, real roots.

- If the discriminant is negative ($b^2 - 4ac < 0$), then the quadratic has no real roots, and two complex roots (roots that are complex numbers). Since complex numbers don't appear in the scope of elementary competition math, we will only deal with the real roots of polynomials.

Hence, the discriminant tells us the number of real roots of a quadratic.

2.5.3 Optimization

Let $p(x) = ax^2 + bx + c$. We can rewrite the quadratic as $a\left(x + \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right]$. If a is positive, then $a\left(x + \frac{b}{2a}\right)^2 \geq 0$. Hence, if $a > 0$, the minimum value of the quadratic is $c - \frac{b^2}{4a}$, which is attained when $a\left(x + \frac{b}{2a}\right)^2 = 0$ or $x = -\frac{b}{2a}$. Similarly, if a is negative, then $a\left(x + \frac{b}{2a}\right)^2 \leq 0$. Thus, if $a < 0$, the maximum value of the quadratic is $c - \frac{b^2}{4a}$, which is attained when $x = -\frac{b}{2a}$. Therefore, when $a \neq 0$, the point $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ is either the maximum or minimum point (depending on the sign of a, as discussed above). We call this point the **vertex** of the parabola.

2.5.4 Graphing Quadratics

A quadratic plotted on the coordinate plane is called a **parabola**. To graph a quadratic, we follow these steps.

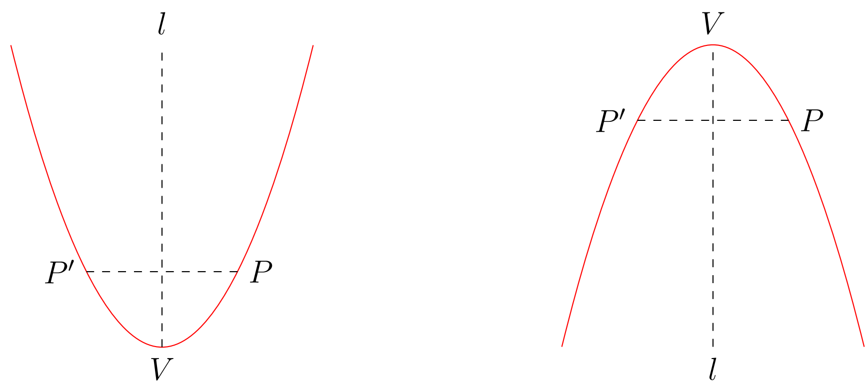
1. Determine the vertex of the parabola.

2. Find a point on the parabola by plugging in a value for x into the quadratic. Notice that by completion of squares, any parabola is symmetric with respect to the vertical line through its vertex. Hence, by finding one point on the parabola, you can reflect it across this vertical line to get another point on the parabola.

3. Depending on the sign of a, determine whether the parabola opens upwards or downwards. That is, if $a > 0$, the parabola opens upwards and if $a < 0$, the parabola opens downwards.

4. If the parabola opens upwards, draw a "U" through the three points with minimum at the vertex. Likewise, if it opens downwards, draw an upside-down "U" through the three points with maximum at vertex.

This process is reflected in the below two diagrams.



The left diagram shows an upward-opening parabola with vertex V passing through point P and point P' (which is the reflection of P over the vertical line l passing through V). Similarly, the right diagram shows a downward-opening parabola with vertex V passing through point P and point P' (which is the reflection of P over the vertical line l passing through V).

2.6 Exercises

1. Find the maximum integer value of a such that $x^2 - 5x + a$ has two real roots.

2. Find the roots of $x + 1/x = 3$.

3. Let r_1 and r_2 be the roots of the quadratic $ax^2 + bx + c$. Find $\frac{1}{r_1} + \frac{1}{r_2}$ in terms of a, b, and c.

4. Find all b such that $x^2 + bx + 9$ has exactly one real root.

5. Let r_1 and r_2 be the roots of the quadratic $ax^2 + bx + c$. Find $|r_1 - r_2|$ in terms of a, b, and c.

6. Let $f(x)$ be the polynomial
$$f(x) = 3x^4 + 5x^2 - 9x - 2.$$
If $g(x)$ is equal to the polynomial $f(x - 1)$, what is the sum of the coefficients of g? (Hint: How can you easily find the sum of the coefficients of any polynomial?)

7. If the parabola $y_1 = x^2 + 2x + 7$ and the line $y_2 = 6x + b$ intersect at only one point, what is the value of b? *Source: AoPS*

8. If r, s, and t are the roots of $x^3 + 4x^2 - 7x + 12$, what is $(r + s)(s + t)(t + r)$?

9. Find the positive integer values of c for which the equation $5x^2 + 11x + c = 0$ has rational solutions. *Source: AoPS*

10. The roots of the quadratic $p(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a? *Source: AMC 12*

11. Solve the following system of equations in real numbers (x, y, z). *Source: 105 Algebra Problems*

$$\begin{cases} 2x - z^2 &= -7 \\ 4y - x^2 &= 7 \\ 6z - y^2 &= 14 \end{cases}$$

12. For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$? *Source: AMC 10*

13. Let the cubic $x^3 - 4x^2 + 5x - 1.9$ have real roots a, b, and c. Find the area of the triangle with sides of length a, b, and c. *Source: Mandelbrot*

14. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. Find the probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2$$

are all real. *Source: AIME*

15. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$? (Hint: Use the Stars and Bars technique) *Source: AMC 10*

16. Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$. *Source: AIME*