Chapter 17

Solutions to Polynomial Problems

17.1 Warm-up Problems

1. We square both sides of the equation to get $19+3y=49 \implies 3y=30 \implies y=\boxed{10}$.

2. When we expand the product using the distributive property, four terms will contain a, four terms will contain b, and four terms will contain c. Thus, there are a total of $3 \cdot 4 = \boxed{12}$ terms in the expansion.

3. (a) We multiply the top equation by -2 to get $-2x-4y=-6$. Adding this to the bottom equation, we get $y=2$. Plugging this back in to the top equation, we find that $x+2\cdot 2=3 \implies x=-1$. Thus, the solution to this system is $\boxed{(-1,2)}$.

   (b) If we multiply the top equation by 3, we get $3x+6y=9$. Note that this line shares the same slope as the bottom equation, but has a different y-intercept. This means they are parallel, and do not intersect. Thus, there are $\boxed{\text{no solutions}}$.

   (c) If we multiply the top equation by 3, we see that the two equations are the same line. This means there are $\boxed{\text{infinitely many solutions}}$.

17.2 Checkpoints

1. Using Vieta's Formula, the sum of the roots is $\frac{-(-5)}{1} = \boxed{5}$.

2. Applying Vieta's again, the product of the roots is $(-1)^5\frac{10}{1} = \boxed{-10}$.

3. We have that

$$
\begin{aligned}
3x^2 - 2x - 1 &= 3\left(x^2 - \frac{2}{3}x\right) - 1 \\
&= 3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3} - 1 \\
&= \boxed{3\left(x - \frac{1}{3}\right)^2 - \frac{4}{3}}.
\end{aligned}
$$

17.3 Long Division with Polynomials Review Exercises

1. We divide $x^4 - 3x^3 + 15x - 25$ by $x^2 - 3x + 5$ using long division:

$$
\begin{array}{r@{}rrrrr}
 & & & x^2 & & -\,5 \\
\cline{2-6}
x^2 - 3x + 5 \big) & x^4 & -\,3x^3 & & +\,15x & -\,25 \\
 & -\,x^4 & +\,3x^3 & -\,5x^2 & & \\
\cline{2-4}
 & & & -\,5x^2 & +\,15x & -\,25 \\
 & & & 5x^2 & -\,15x & +\,25 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

Thus $p(x) = \boxed{x^2 - 5}$.

2. We use long division:

$$
\begin{array}{r@{}rrrrrrr}
 & & x^5 & -\,x^4 & +\,x^3 & -\,x^2 & +\,x & -\,1 \\
\cline{2-8}
x + 1 \big) & x^6 & & & & & & -\,3 \\
 & -\,x^6 & -\,x^5 & & & & & \\
\cline{2-3}
 & & -\,x^5 & & & & & \\
 & & x^5 & +\,x^4 & & & & \\
\cline{3-4}
 & & & x^4 & & & & \\
 & & & -\,x^4 & -\,x^3 & & & \\
\cline{4-5}
 & & & & -\,x^3 & & & \\
 & & & & x^3 & +\,x^2 & & \\
\cline{5-6}
 & & & & & x^2 & & \\
 & & & & & -\,x^2 & -\,x & \\
\cline{6-7}
 & & & & & & -\,x & -\,3 \\
 & & & & & & x & +\,1 \\
\cline{7-8}
 & & & & & & & -\,2
\end{array}
$$

So the quotient is $x^5 - x^4 + x^3 - x^2 + x - 1$ and we have a remainder of -2.

17.4 Roots Review Exercises

1. We can factor $c^2+6c-27$ as $(c-3)(c+9)$. Thus, we can cancel the two $(c-3)$ in the fraction to get the equation $c+9+2c=23$. Solving for c, we get $c = \boxed{\dfrac{14}{3}}$.

2. You can notice that this is a 6-8-10 Pythagorean triple, so x can equal either 8 or -8 for a sum of 0. Alternatively, we can use Vieta's. However, since $b=0$ in this equation the sum of the roots is just $\boxed{0}$.

3. We can substitute -2 for y to get the quadratic $-2 = x^2-7x+7$. However, note that we're looking for the sum of the x-coordinates, or the sum of the roots of the quadratic in other words. According to Vieta's, the sum of the roots will be $-(-7)/1 = \boxed{7}$.

4. Let a and b be the roots of $x^2+13x+6$. Then $(a+b) = -13$ from Vieta's, and $(a+b)^2 = a^2+2ab+b^2 = 169$. In addition, $ab=6$ from Vieta's. Then $a^2+b^2 = a^2+2ab+b^2-2ab = 169-2\cdot 6 = \boxed{157}$.

5. Let a, b, and c be the roots of x^3+5x^2+6x+9. By Vieta's, $a+b+c=-5$ so $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac = 25$. Also by Vieta's, we know $ab+bc+ac=6$ (the coefficient of the x term divided by the coefficient of the x^3 term). Then

$$a^2+b^2+c^2 = a^2+b^2+c^2+2ab+2bc+2ac-2(ab+bc+ac) = 25-2\cdot 6 = \boxed{13}$$

17.5 Problems

1. For the polynomial to have two real roots, the discriminant must be positive. Hence, $(-5)^2-4(1)(a) > 0$ or $a < \frac{25}{4}$. Therefore, the maximum integer value of a such that the polynomial has two real roots is $\lfloor \frac{25}{4} \rfloor = \boxed{6}$.

2. To get rid of the fraction $\frac{1}{x}$, we multiply both sides of the equation by x and rearrange to get $x^2-3x+1=0$. Using the quadratic formula, we get that the roots are $\boxed{\dfrac{3+\sqrt{5}}{2}}$ and $\boxed{\dfrac{3+\sqrt{5}}{2}}$.

3. By getting a common denominator, we have that $\frac{1}{r_1}+\frac{1}{r_2} = \frac{r_1+r_2}{r_1r_2}$. By Vieta's, we know that $r_1+r_2-\frac{b}{a}$ and $r_1r_2 = \frac{c}{a}$. Hence, $\frac{1}{r_1}+\frac{1}{r_2} = \boxed{-\dfrac{b}{c}}$.

4. For the polynomial to have exactly one real root, the discriminant must be zero. Hence, $b^2-4(1)(9)=0$ and $b=\boxed{6}$ or $\boxed{-6}$.

5. The roots of ax^2+bx+c are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$. $|r_1-r_2|$ is equivalent to the distance between the two roots, which is

$$\frac{-b+\sqrt{b^2-4ac}}{2a}-\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{2\sqrt{b^2-4ac}}{a}=\boxed{\frac{\sqrt{b^2-4ac}}{a}}$$

6. A key observation (which is very useful in competition math) is that the sum of the coefficients of any polynomial p is given by $p(1)$. Thus, the sum of the coefficients of g is $g(1)=f(1-1)=f(0)=\boxed{-2}$.

7. If the parabola and the line intersect at (x,y), then $x^2+2x+7=6x+b$ or $x^2-4x+7-b=0$. Since they intersect at exactly one point, the discriminant to the above quadratic must be 0. Thus, $(-4)^2-4(1)(7-b)=0$, so $b=\boxed{3}$.

8. Let $p(x)$ be the polynomial. By Vieta's, $r+s+t=-4$. We have

$$\begin{aligned}(r+s)(s+t)(t+r)&=([r+s+t]-t)([r+s+t]-r)([r+s+t]-s)\\&=(-4-t)(-4-r)(-4-s)\\&=p(-4)\\&=(-4)^3+4(-4)^2-7(-4)+12\\&=\boxed{40}.\end{aligned}$$

9. According to the Quadratic Formula, the roots of $5x^2+11x+c$ are $\frac{-11\pm\sqrt{11^2-4\cdot5\cdot c}}{2\cdot5}$. If we want the roots to be rational solutions, then the discriminant, $11^2-4\cdot5\cdot c=121-20c$ must be a perfect square since we are given that c is a positive integer. $121-20c$ is a perfect square only when $c=\boxed{2}$ and $c=\boxed{6}$.

10. Let r_1 and r_2 be the roots of $x^2-ax+2a$. From Vieta's, the sum of the roots is $r_1+r_2=-(-a)=a$ and the product of the roots is $r_1r_2=2a$. We multiply the first equation by 2 to get $2r_1+2r_2=2a$, so that means $2r_1+2r_2=r_1r_2$. We rearrange this equation and do some manipulations.[1]

$$\begin{aligned}&2r_1+2r_2=r_1r_2\\\implies\quad&r_1r_2-2r_1-2r_2=0\\\implies\quad&r_1r_2-2r_1-2r_2+4=4\\\implies\quad&(r_1-2)(r_2-2)=4\end{aligned}$$

Since the roots are integers, (r_1-2) and (r_2-2) must be integer factors of 4: -1 and -4, -2 and -2, 1 and 4, and 2 and 2. This gives us the pairs (1,-2), (0,0), (-2,1), (3,6), (4,4), and (6,3). This is turn gives us the values of -1, 0, 8, and 9 for a with a total sum of $\boxed{16}$.

[1] The factorization trick used below is known as Simon's Favorite Factoring Trick.

11. We sum the equations and complete the square.

$$\begin{aligned}\left(2x - z^2\right) + \left(4y - x^2\right) + \left(6z - y^2\right) &= \left(-x^2 + 2x\right) + \left(-y^2 + 4y\right) + \left(-z^2 + 4z\right)\\ &= -\left[(x-1)^2 - 1 + (y-2)^2 - 4 + (z-3)^2 - 9\right]\\ &= -\left[(x-1)^2 + (y-2)^2 + (z-3)^2 - 14\right]\\ &= -7 + 7 + 14\\ &= 14.\end{aligned}$$

Therefore, $(x-1)^2 + (y-2)^2 + (z-3)^2 = 0$. We have that $x - 1 = 0$, $y - 2 = 0$, and $z - 3 = 0$, so the only solution to the system is $\boxed{x = 1}$, $\boxed{y = 2}$, and $\boxed{z = 3}$.

12. Because $f(x)$ has a degree of 4, we know it has 4 roots, and we can say the same for $g(x)$, which has 3 roots. Because we know $g(x)$ divides $f(x)$, and the leading coefficient of both polynomials are the same, we can write $f(x)$ and $g(x)$ as

$$f(x) = g(x)(x - r),$$

where r is the fourth root of $f(x)$. We can now substitute g(x) into the equation to get

$$\begin{aligned}f(x) &= (x^3 + ax^2 + x + 10)(x - r)\\ &= x^4 + (a - r)x^3 + (1 - ar)x^2 + (10 - r)x - 10r.\end{aligned}$$

We can now compare the coefficients of f(x) from this equation to our given function to create a system of equations. Doing so would result in

$$\begin{aligned}a - r &= 1\\ 1 - ar &= b\\ 10 - r &= 100\\ -10r &= c\end{aligned}$$

From this system we can obviously find that $r = -90$ with the third equation, and we can find that $c = -900$ with the fourth equation with the value r. With the first equation, we find that $a = -89$, so with the second equation we have $1 - (-89)(-90) = b = -8009$.

Since we now know b and c, we can plug those two numbers into the polynomial and get $f(x) = x^4 + x^3 - 8009x^2 + 100x + 900$, so we can plug $x = 1$ to get $f(x) = 1 + 1 - 8009 + 100 + 900 = \boxed{-7007.}$

13. Let $s = \frac{a+b+c}{2}$. Using Heron's Formula, our desired area is given by $\sqrt{s(s-a)(s-b)(s-c)}$. To evaluate this, we first compute s. By Vieta's, $a + b + c = 4$ so $s = \frac{4}{2} = 2$. Now, notice that

$$\begin{aligned}(s-a)(s-b)(s-c) &= f(s)\\ &= s^3 - 4s^2 + 5s - 1.9\\ &= .1\end{aligned}$$

Thus, our area is $\sqrt{2 \cdot 0.1} = \boxed{\sqrt{0.2}}$.

14. First, let's try to factor this polynomial. The Rational Root Theorem tells us that $x = -2, -1, 1, 2$ are possible rational roots for this polynomial. We can use polynomial long division or synthetic division to check the possible roots. Either way, we find that $x = -2$ and $x = 1$ are factors of the polynomial, and it factors as $(x + 2)(x - 1)(x^2 + (2a - 1)x + 1)$.

    For all the roots of $(x^2 + (2a - 1)x + 1)$ to be real, its discriminant must be greater than or equal to zero: $(2a - 1)^2 - 4 \cdot 1 \cdot 1 \geq 0 \implies 4a^2 - 4a - 3 \geq 0$. It turns out that we can factor this as $(2a - 3)(2a + 1) \geq 0$. To find the solution set of this inequality, we plug in 0 for a to get $(-3)(1) \ngeq 0$. Thus 0 is not in the solution set. This means $a \leq -\frac{1}{2}$ or $a \geq \frac{3}{2}$.

    We want to find the probability that a satisfies the inequality within the range $[-20, 18]$. This is a geometric probability problem: the probability is equivalent to the length of the number line where $a \leq -\frac{1}{2}$ or $a \geq \frac{3}{2}$ over the length of the range, which is $\frac{(-\frac{1}{2}-(-20))+(18-\frac{3}{2})}{18-(-20)} = \boxed{\frac{18}{19}}$.

15. * Let $p(x) = ax^3 + bx^2 + cd + d$ be one such polynomial (i.e, its coefficients belong to the set $\{0, 1, \ldots, 0\}$ and $p(-1) = -9$. We have that $p(-1) = -a + b - c + d = -9$. Now we can cleverly apply the Stars and Bars technique by substituting $x = 9 - a$ and $y = 9 - b$ to get the equation $x + y + c + d = 9$. Notice that like a, b, c and d, x and y must be integers between 0 and 9 inclusive. We have a total of $9 + 4 - 1 = 12$ "slots" to place the "bars" and $4 - 1 = 3$ "bars", so our answer is $\binom{12}{3} = \boxed{220}$ (refer to the Stars and Bars chapter for a generic formula and an in depth explanation for similar sub-problems).

16. We using long division, we divide $P(x)$ by $Q(x)$. This gives us $\frac{P(x)}{Q(x)} = x^2 + 1$ with a remainder of $x^2 - x + 1$. Thus, $P(x) = Q(x)(x^2 + 1) + x^2 - x + 1$. Since z_1 is a root of $Q(x)$, $P(z_1) = z_1^2 - z_1 + 1$ (and likewise for all the other roots of $Q(x)$. This means

$$P(z_1) + P(z_2) + P(z_3) + P(z_4) = (z_1^2 + z_2^2 + z_3^2 + z_4^2) - (z_1 + z_2 + z_3 + z_4) + 4.$$

    From Vieta's, we know $(z_1 + z_2 + z_3 + z_4) = 1$. To find $(z_1^2 + z_2^2 + z_3^2 + z_4^2)$, we square $(z_1 + z_2 + z_3 + z_4)$ to get

$$z_1^2 + 2z_2z_1 + 2z_3z_1 + 2z_4z_1 + z_2^2 + z_3^2 + z_4^2 + 2z_2z_3 + 2z_2z_4 + 2z_3z_4 = 1^2$$

    which equals

$$(z_1^2 + z_2^2 + z_3^2 + z_4^2) + 2(z_1z_2 + z_1z_3 + \cdots) = 1$$

    From Vieta's, we know the pairwise sum of the roots is -1, so $(z_1^2 + z_2^2 + z_3^2 + z_4^2) + 2(-1) = 1 \implies (z_1^2 + z_2^2 + z_3^2 + z_4^2) = 3$.

    Thus, our overall sum will be $3 - 1 + 4 = \boxed{6}$.