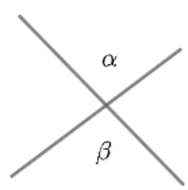
Iowa City Math Circle Handouts

July 28, 2019

8 Geometry Techniques

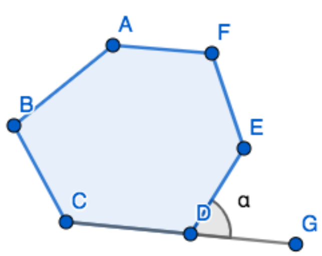
8.1 Warm-up Problems

1. Why does $\alpha = \beta$ in the figure below? These are called vertical angles.



2. What is the sum of the angles inside a polygon with n sides, given the sum of the angles in a triangle is $180°$? Why?

3. $\angle EDG$ in the figure below is an example of an exterior angle of a polygon.

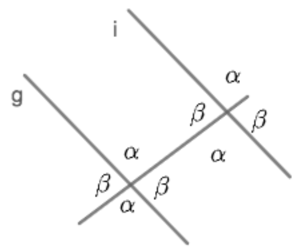


For a n-gon, what is the sum of all of its exterior angles?

8.2 Angle Chasing

Angle chasing is the art of using geometric tools to find the measures of angles in a figure.

Parallel lines are one of the main tools we can use. If a line intersects a pair of parallel lines, then there are several pairs of congruent angles:



All angles marked with α are congruent, and all angles marked with β are congruent. In addition, if any angles are congruent in the configuration above, then we know that the two lines are parallel.

Checkpoint 8.1. Using this knowledge, prove that the sum of the angles in a triangle is 180°.

Besides angle chasing, there are several definitions and notations that will be helpful.

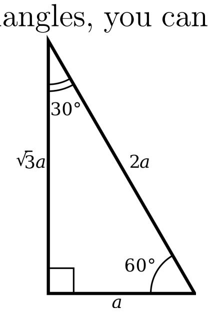
Definition. An angle bisector is a line through an angle that splits it into two congruent angles.

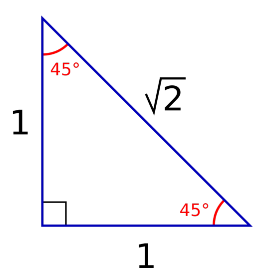
Definition. A right angle is a 90° angle. A right triangle is a triangle that contains a right angle.

Definition. Two lines are perpendicular if their intersection forms a right angle. If line k is perpendicular to line l, we can write $k \perp l$. Furthermore, a line is a perpendicular bisector if its intersection with another line segment forms a right angle and splits the segment into two equal lengths.

Definition. A triangle is isosceles if it has two sides with the same length, or two angles that are congruent. If either of these conditions are fulfilled, the triangle is isosceles (we'll explore why in the similarity section). Be on the lookout for these!

Definition. 30-60-90 and 45-45-90 have special properties. If you see one of these triangles, you can calculate their side lengths easily because they are in a specific ratio.





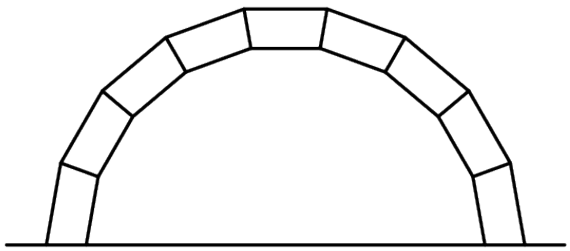
The most helpful thing you can do to solve angle chasing problems (or any geometry problem) is to draw a large, neat diagram! This way, you'll be able to see the relationships between different angles better. We highly recommend getting a straightedge or a ruler to help keep your lines straight.

You should also be marking any congruent angles or segments. This will also help you quickly see relationships in the diagram. Try drawing a diagram in each of the following review exercises.

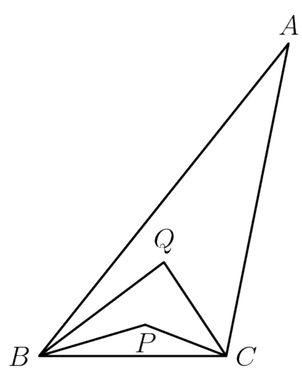
8.2.1 Review Exercises

1. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD? *Source: AMC 12*

2. The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?

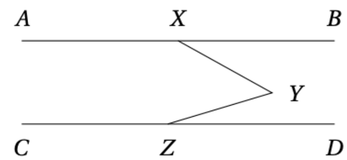


3. The trisectors of angles B and C of scalene triangle ABC meet at points P and Q as shown. Angle A measures 39 degrees and angle QBP measures 14 degrees. What is the measure of angle BPC? *Source: MATHCOUNTS*



4. Let $ABDC$ be a quadrilateral with $AB = BC = CD$, $\angle B = 100°$ and $\angle C = 130°$. Find $\angle A - \angle D$.

5. In the figure below, $\overline{AB} || \overline{CD}$, $\angle BXY = 45°$, $\angle DZY = 25°$, and $XY = YZ$. What is the degree measure of $\angle YXZ$? *Source: Purple Comet*



6. Let ABC be a triangle with $AB = AC$ and let K and M be points on the side AB and L a point on the side AC such that $BC = CM = ML = LK = KA$. Find $\angle A$. *Source: 106 Geometry*

7. The degree measures of the angles of a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle. *Source: AIME*

8. * In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. Find the area of triangle ABC. (Hint: Draw the altitude of ΔABC from C) *Source: AIME*

8.3 Similarity and Congruence

Similarity and congruence are also two very powerful tools for angle chasing. Two triangles are similar when the dilation of one of the triangles is congruent to the other. In other words, two triangles are similar when they have the same angles and their corresponding sides are in proportion. However, we don't need to check that all sides are in proportion and that the angle measures are equal. There are special rules we can use to quickly tell whether triangles are congruent.

In addition, two triangles are congruent if they are the "same" - more technically, they have the same angle measures and same side lengths. This means that if two triangles are similar and have side lengths in a ratio of 1:1, then there are congruent. As a result, we can use a few special variations of the similarity rules to determine congruence.

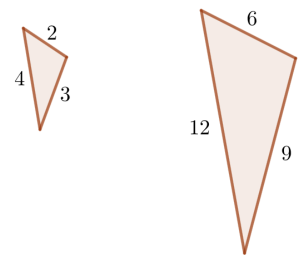
8.3.1 SSS Similarity and Congruence

Two triangles are similar if their corresponding sides are in proportion. This means that if

$$\frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

then ΔABC is similar to ΔDEF.

The following two triangles are similar from SSS Similarity:



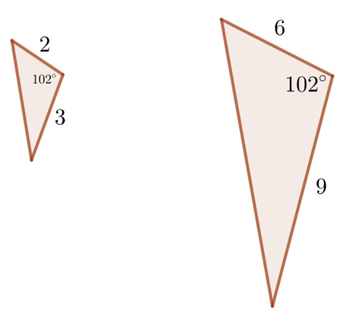
Likewise, SSS Congruence says that if two triangles have the same side lengths, then they are congruent. In other words, having the same side lengths implies that the two also have the same angle measures, hence, they are congruent.

The following two triangles are congruent by SSS Congruence:

8.3.2 SAS Similarity and Congruence

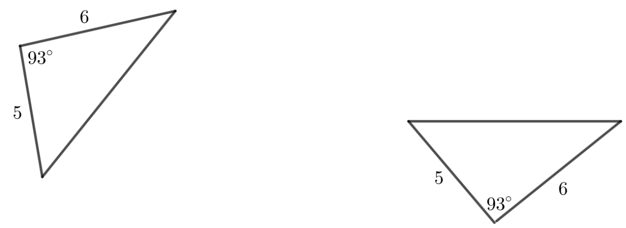
Two triangles are similar if two sets of sides are in proportion, and the angle between them are equal in the two triangles.

The following two triangles are similar from SAS Similarity:



SAS Congruence says that if two triangles have two sides that are the same length and the angle between the two sides are equal, then the two triangles are congruent.

The following triangles are congruent by SAS Congruence:

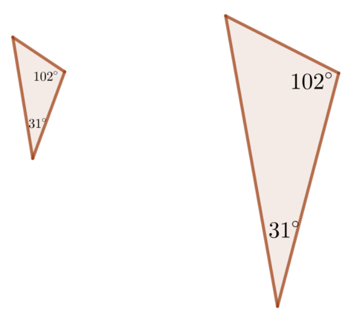


8.3.3 AAA (or AA) Similarity

Two triangles are similar if they have the same angles. However, if we know two angles in a triangle, we know the third: x, y, and $180 - x - y$. Hence, we only need that the two triangles have two angles in common (AA).

There is no AA congruence because the side lengths of the similar triangles can differ.

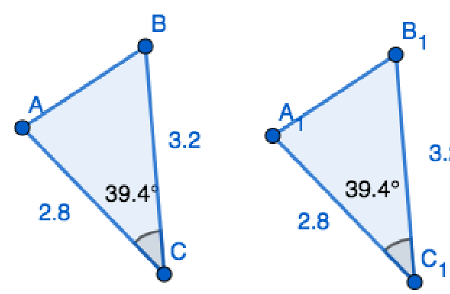
The following two triangles are similar from AA Similarity:



8.3.4 ASA Congruence

If two triangles have two angles that are congruent and if the sides in between the two angles have the same length, the two triangles are congruent.

The following pair of triangles are congruent by ASA Congruence:

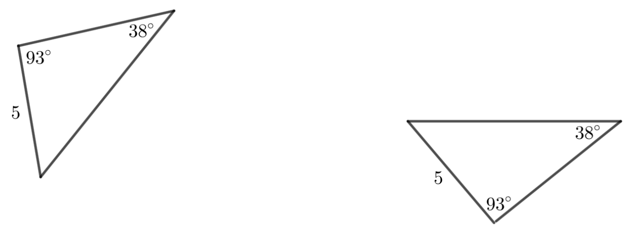


Note that ASA Similarity is the same as AA Similarity, because we have two congruent angles.

8.3.5 SAA Congruence

SAA Congruence says that if two triangles have two angles and a side length in common, then they are congruent. The common side must be in the same position relative to the two common angles.

The following pair of triangles are congruent by SAA Congruence:



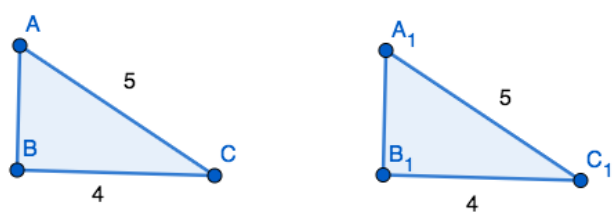
SAA Congruence is equivalent to ASA Congruence because you can find the measure of the other angle adjacent to the side length. We actually recommend ASA Congruence over SAA Congruence because SAA Congruence can get confusing.

Checkpoint 8.2. The following triangles are NOT congruent - why?

Again, SAA Similarity is the same as AA Similarity because we know that two of the angles are congruent.

8.3.6 HL Congruence

Two right triangles are congruent if they have a congruent hypotenuse and leg.



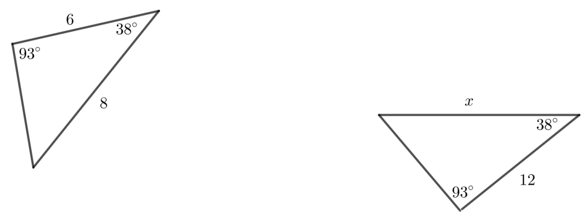
For example, if $\angle ABC = \angle A_1B_1C_1 = 90°$, then ΔABC and $\Delta A_1B_1C_1$ are congruent by HL congruence because their hypotenuses are both 5, and one of their legs are 4.

If two right triangles' hypotenuse and leg are in proportion, then they will be similar. However, HL Similarity isn't as useful as HL Congruency.

Checkpoint 8.3. Why is HL congruence true?

8.4 Review Exercises

1. Find x in the figure below:



2. In triangle ABC, M is the midpoint of AB and N is the midpoint of AC. Prove that the length of MN is $\frac{1}{2}BC$.

3. Let $ABCD$ be a parallelogram. Show that ΔABC is congruent to ΔCDA using SSS, SAS, and SAA Congruence.

4. The dimensions of a triangle are tripled to form a new triangle. If the area of the new triangle is 54 square feet, how many square feet were in the area of the original triangle? *Source: Alcumus*

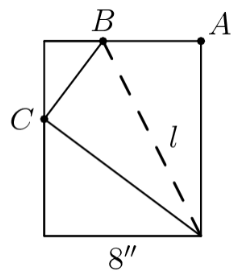
5. Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. What is the ratio of the area of the other small right triangle to the area of the square? Express your answer as a common fraction in terms of m. *Source: Alcumus*

6. Right triangle ABC has one leg of length 6 cm, one leg of length 8 cm and a right angle at A. A square has one side on the hypotenuse of triangle ABC and a vertex on each of the two legs of triangle ABC. What is the length of one side of the square, in cm? Express your answer as a common fraction. *Source: Alcumus*

7. Let ΔABC be a triangle with $\angle A = 70°$. In addition, let D be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Given that $\angle ACD = 95°$ and $\angle ABD = 85°$, find the measure of $\angle ACB$.

8. Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$? *Source: AMC 10*

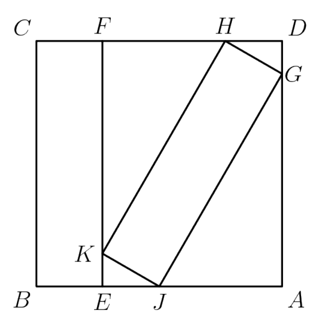
9. Corner A of a rectangular piece of paper of width 8 inches is folded over so that it coincides with point C on the opposite side. If $BC = 5$ inches, find the length in inches of fold l.



Source: Alcumus

10. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$? *Source: AMC 10*

11. In the figure, $ABCD$ is a square of side length 1. The rectangles $JKHG$ and $EBCF$ are congruent. What is BE? *Source: AMC 12*



Contributors: Ananth Shyamal, Divya Shyamal, Reece Yang, Daniel Stewart, Kevin Yang
iowacitymathcircle.org