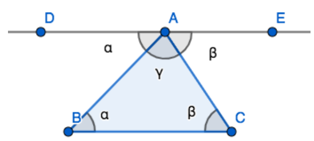
# Summer Math Circle Handouts

July 28, 2019

## 8 Solutions to Geometry Technique Problems

### 8.1 Checkpoints

1. *Proof.* We draw a parallel line through one of the vertices of the triangle:




Since $DE$ is parallel to $BC$, $\angle DAB \cong \angle ABC$ and $\angle EAC \cong \angle ACB$. This means the angle measures of the triangle lie on a straight line, which implies that $\alpha + \beta + \gamma = 180°$, and we are done! □

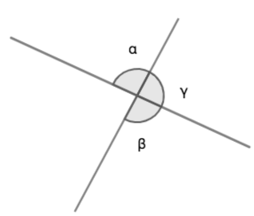
2. Note that the side with length 6 is opposite the 93° angle in the first triangle, while in the second triangle the side with length 6 is in between the two given angles. Since the positions of the sides and angles do not match up, these triangles are not congruent.

   In cases like this, we highly recommend finding the measure of the third angle to clarify whether the triangles are actually similar (you can often get confused, especially if your diagram is very complex).

3. We can use the Pythagorean theorem to find the length of the other leg. Since we use all the same values, we'll get the same length for the leg. The triangles are then congruent by SSS Congruence.

### 8.2 Warm-up Problems

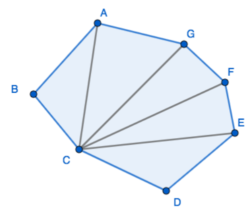
1. *Proof.* Let $\gamma$ be the angle adjacent to $\alpha$ and $\beta$ in the figure below.



Because $\alpha$ and $\gamma$ form a straight angle, $\alpha+\gamma = 180°$. Likewise, $\beta+\gamma = 180° \implies \gamma = 180° - \beta$. We can substitute this in for $\gamma$ to get $\alpha + 180° - \beta = 180° \implies \alpha = \beta$. □

2. The sum of the angles in a $n$-gon will be $(n-2)180°$.

   *Proof.* We can split the polygon into $n-2$ triangles by drawing line segments from a point, like in the figure below:

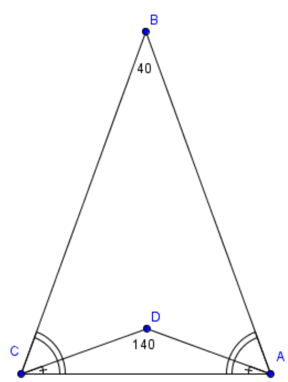



   Since we the sum of each of the angles in all the triangles add up to the sum of the interior angles of the polygon, the total sum is simply $(n-2)180°$. □

3. Each exterior angle is equivalent to 180° minus its adjacent interior angle. Thus, the sum of the exterior angles will be equivalent to $180°n$ minus the sum of the interior angles, which we know is $(n-2)180°$ from the previous problem. This gives us the expression $180°n - (n-2)180°$ which evaluates to $\boxed{360°}$.

## 8.3 Angle Chasing Review Exercises

1. First, draw a diagram!



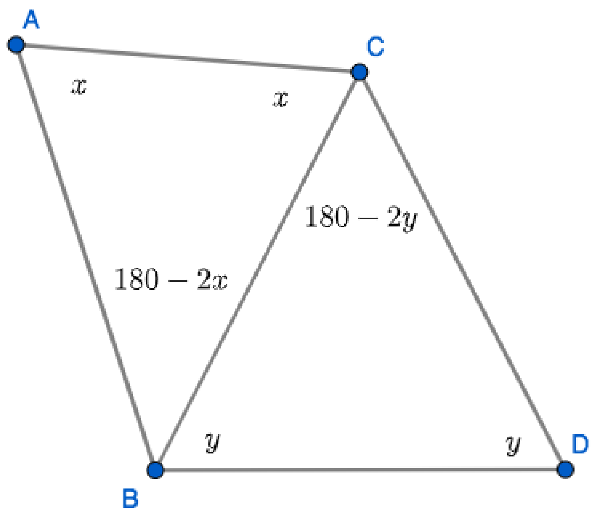

   *Credit: AoPS Wiki*

   Since $\Delta ABC$ is isoceles, we know $\angle BAC = \angle BCA = \frac{180-40}{2} = 70°$. Likewise, $\angle DAC = \frac{180-140}{2} = 20°$. So, $\angle BAD = 70 - 20 = \boxed{50°}$.

2. You can extend the legs of the isosceles trapezoid to the center of the 18-gon. Since there are 9 isosceles triangles against a straight line, each vertex angle of the triangles will have degree $\frac{180}{9} = 20$. The remaining base angles of the triangles will have degree $180 - x$. This gives us the equation $180 - x + 180 - x + 20 = 180 \implies x = \boxed{100^\circ}$

3. Since $\angle ABC = 3\angle QBP = 3 \cdot 14 = 42$, we know $\angle ACB = 180 - 39 - 42 = 99$. Then $\angle BCP = \frac{1}{3} \cdot 99 = 33$. $\angle PBC$ also equals 14, so $\angle BPC = \boxed{133^\circ}$.

4. Be careful here; note that the quadrilateral is $ABDC$, not $ABCD$. Drawing our diagram, we find two isosceles triangles:

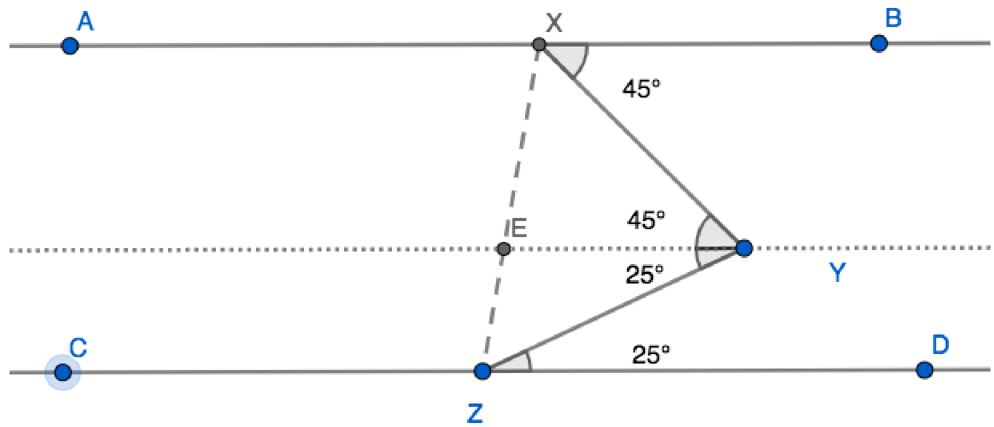



Let $\angle A = x$ and $\angle D = y$. Then we get that $\angle BCD = 180 - 2y$ and $\angle ABC = 180 - 2x$. Then we get the system of equations:

$$\begin{cases} x - 2y + 180 = 130 \\ -2x + y + 180 = 100 \end{cases}$$

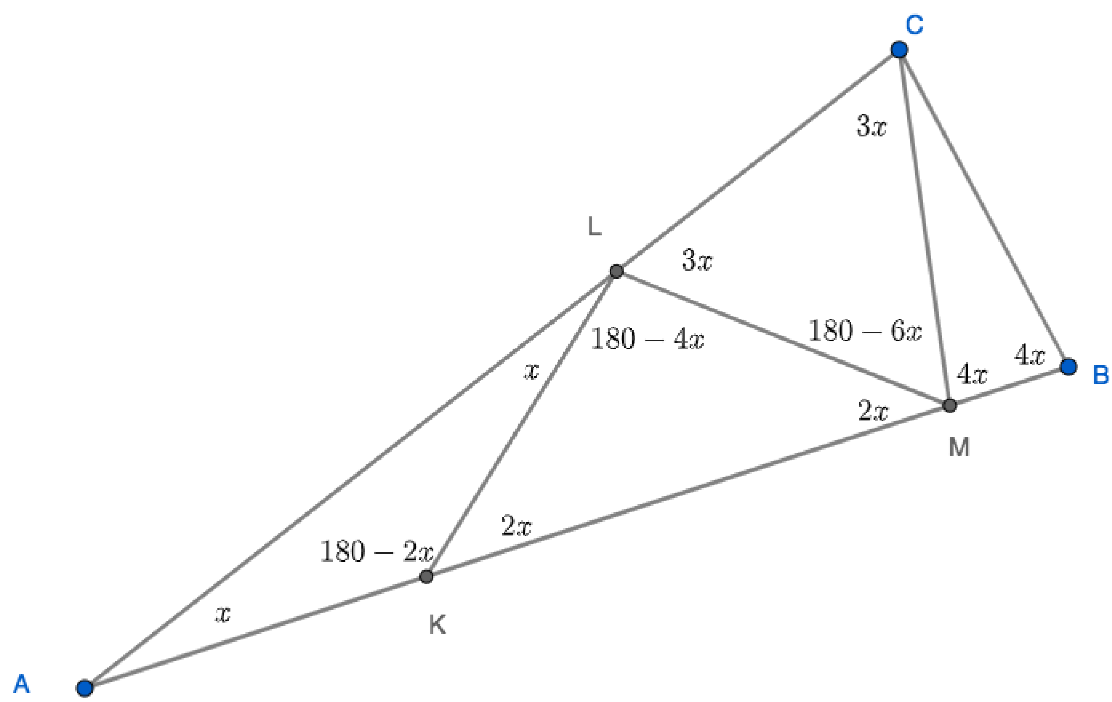
Solving gives us $x = 70$ and $y = 60$, so $\angle A - \angle D = 70 - 60 = \boxed{10^\circ}$.

5. We draw a line parallel to $AB$ and $CD$ through $Y$.




Now we know that $\angle ZYE = 25$ and $\angle XYE = 45$ through the parallel lines. Since $\Delta XYZ$ is isosceles, $\angle YXZ = \frac{180-(25+45)}{2} = \boxed{55°}$.

6. This problem depends on drawing a good diagram and doing lots of angle chasing. If you let $\angle A = x$, you should obtain a diagram similar to the figure below:



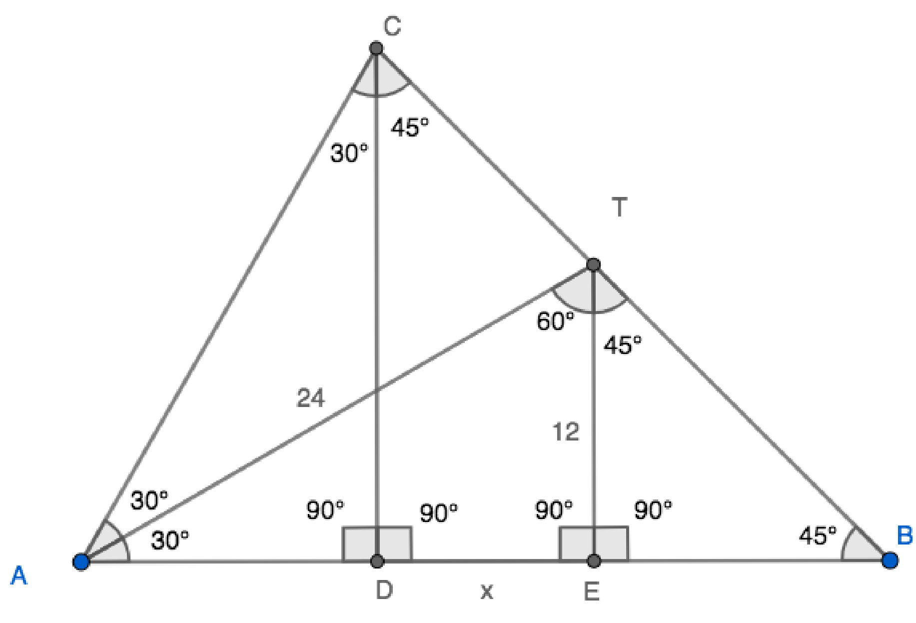

Using the fact that many of the triangles are isosceles, we eventually find that $\angle B = 4x$. Since triangle $ABC$ is isosceles, $\angle B$ also equals $180-2x$. This gives us the equation $\frac{180-x}{2} = 4x \implies x = \boxed{20°}$.

7. We know the sum of all the angles will be $(18-2)180$. Let the measure of the smallest angle be $a$, and let the common difference of the arithmetic series be $d$. The sum of all 18 angles will be $a+(a+d)+(a+2d)+\cdots+(a+17d) = 18a+\frac{(1+17)17}{2}d=(18-2)180$. Simplifying, we get

$$\begin{aligned} 18a+\frac{(1+17)17}{2}d &= (18-2)180 \\ a+\frac{17}{2}d &= 16\cdot 10 \\ 2a+17d &= 320 \end{aligned}$$

We should maximize $a$ and minimize $d$. Otherwise, some angles may be greater than $180°$. So, we let $d=2$ and $a=\boxed{143}$. (You can verify that this works, and that it is the only solution.)

8. A good diagram is key to solving this problem. Make sure you carefully mark all angles, like in the figure below:




Let $DE=x$ like in the figure. We know $AE=12\sqrt{3}$ from 30-60-90 triangle $ATE$. Then from 30-60-90 triangle $ACD$, we know $CD=(12\sqrt{3}-x)\sqrt{3}$. From 45-45-90 triangle $BCD$, we know $CD=12-x$. This gives us the equation $(12\sqrt{3}-x)\sqrt{3}=12-x$. Solving, we get $x=12\sqrt{3}-12$.

The problem asks us to find the area of $ABC$. We now know $CD=x+12=12\sqrt{3}-12+12=12\sqrt{3}$, so the area of $ABC$ is $\frac{1}{2}\cdot 12\sqrt{3}(12\sqrt{3}+12)=\boxed{216+72\sqrt{3}}$.

## 8.4 Similarity and Congruence Review Exercises

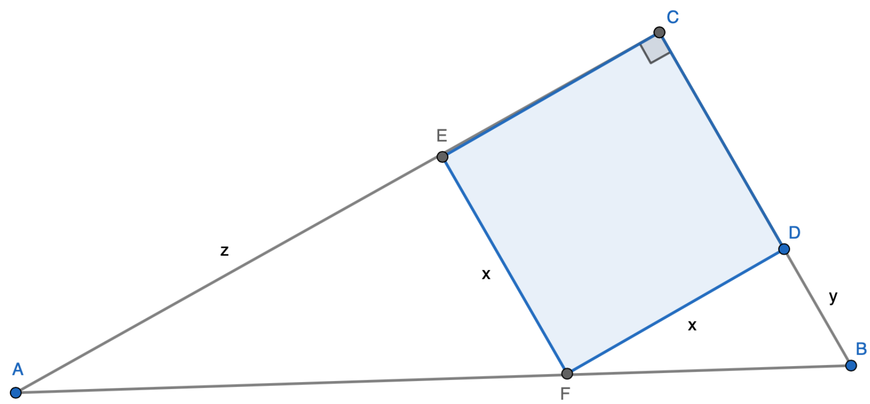
1. Notice that by ASA similarity, the two triangles are similar. We have that $\frac{12}{6} = 2$ is the similarity ratio, so $x = 2 \cdot 8 = \boxed{16}$.

2. The key realization is that $\Delta AMN$ is similar to $\Delta ABC$ in a $1:2$ ratio by SAS similarity. Hence, we get the desired result by using this similarity ratio to $\overline{MN}$.

3. We know that $AB = CD$ and $BC = DA$, and since triangles $\Delta ABC$ and $\Delta CDA$ share diagonal $\overline{AC}$, these triangles are similar by SSS. Similarly, using the fact that $AB \parallel CD$ and $BC \parallel DA$ (and establishing congruent angles by treating diagonal $\overline{AC}$ as a transversal) gives us that these two triangles are also similar by SAS and SAA.

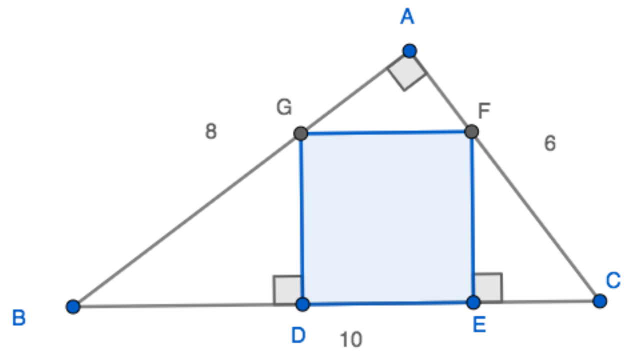
4. Let $b$ and $h$ be base and height of the original triangle, respectively. Then $3b$ and $3h$ are the respective base and height of the new triangle. Using the formula for the area of a triangle ($A = \frac{bh}{2}$), we see that the new triangle has 9 times the area of the original area. Hence, the area of the original triangle was $\frac{54}{9} = \boxed{6}$ square feet.

5. Let's work with the diagram below.



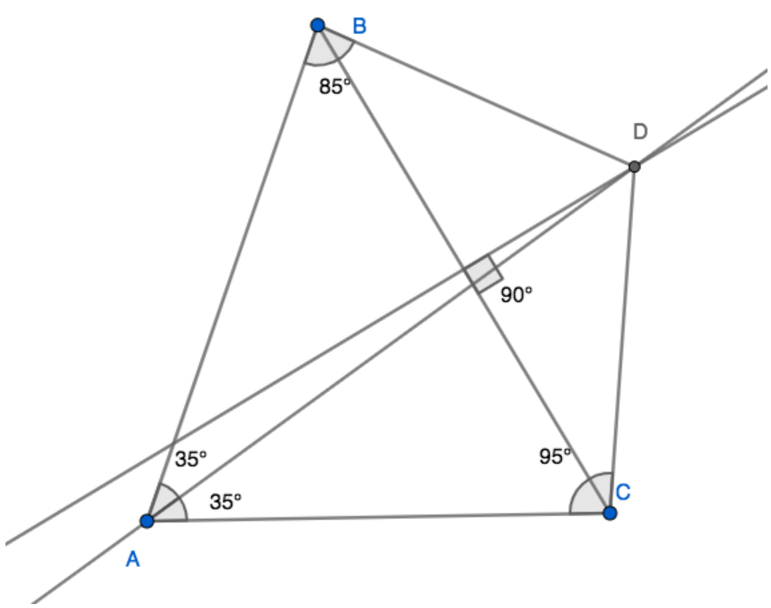

In addition to the diagram, let $[\Delta BFD] = m[CDFE]$, which implies that $\frac{xy}{2} = mx^2$ or $\frac{y}{2} = mx$. Now, the ratio of the area of the other smaller right triangle to the area of the square is $\frac{[\Delta AEF]}{[CDEF]} = \frac{\frac{xz}{2}}{x^2} = \frac{z}{2x}$. Now, we see that $\Delta AEF$ is similar to $\Delta FDB$ by AAA, so $\frac{z}{x} = \frac{x}{y}$. Hence, $z = \frac{x^2}{y}$. Thus, our desired ratio is equivalent to $\frac{x}{2y} = \frac{\frac{y}{2m}}{2y} = \boxed{\frac{1}{4m}}$.

6. Let the side length of the square be $s$.



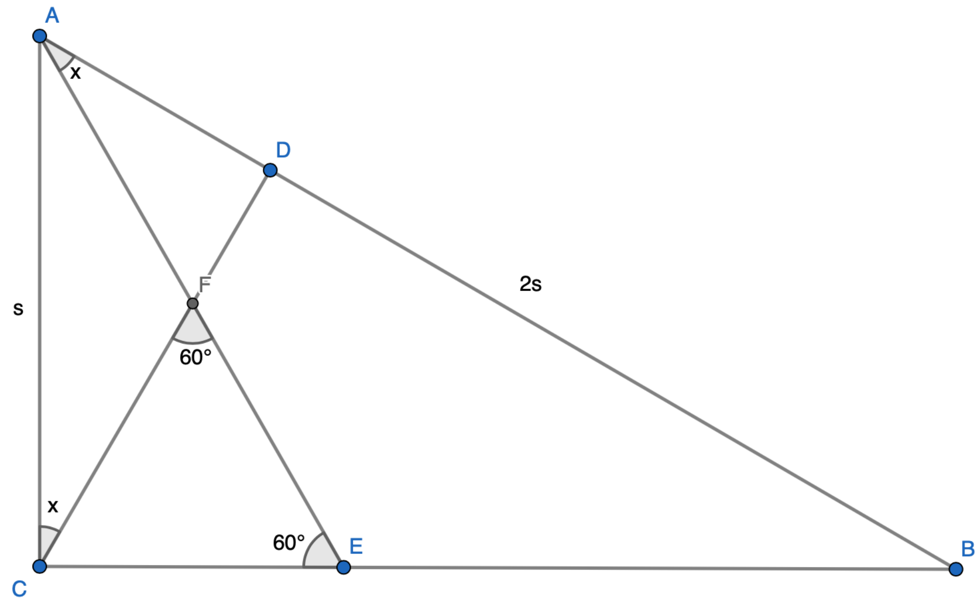

Triangle $AGF$ is similar to triangle $ABC$. We know $GF = s$, so by similarity $AG = \frac{4s}{5}$. Likewise, triangle $BDG$ is similar to triangle $BAC$, so by similarity $BG = \frac{5s}{3}$. Since $BG + AG = AB$, we know $\frac{5s}{3} + \frac{4s}{5} = 8$. Solving, we get $s = \boxed{\dfrac{120}{37}}$

7. First note that $D$ is outside of triangle $ABC$:




Since $D$ is is on the perpendicular bisector of $BC$, this means triangle $BCD$ is isosceles (because the two triangles formed by splitting $\Delta BCD$ with the perpendicular bisector are congruent by HL Congruency). In addition, $\angle BDC = 360 - 70 - 85 - 95 = 110$. Thus, $\angle DCB = \frac{180-110}{2} = 35$. So, $\angle ACB = 95 - 35 = \boxed{60^\circ}$.

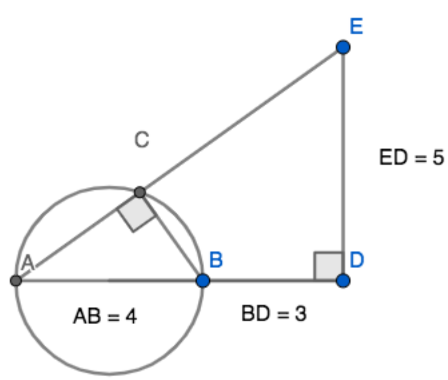
8. The following diagram reflects all the information presented in the problem.




By angle chasing, we see that $\angle AFC = 120°$, $\angle FAC = 60 - x°$, $\angle AEB = 120°$, and $\angle ABE = 60-x°$. Hence, $\Delta AFC$ is similar to $\Delta BEA$ by AAA. Since $\frac{AC}{AB} = \frac{1}{2}$, we have that $\frac{FC}{EA} = \frac{1}{2}$. Furthermore, $FC = FE$ (using the fact that $\Delta CFE$ is isosceles), we get $FC = FA$. Thus, $\Delta AFC$ is isosceles and $x = \angle CAF = \angle ACF = 30°$. Therefore, $\angle ACB = \boxed{90°}$.

9. Let the top left corner of the paper be $D$, the bottom left corner be $E$, and the bottom right corner be $F$. Since the paper is being folded, we know $BC = BA = 5$. As a result, $DE = 8 - 5 = 3$. This means $BDC$ is a 3-4-5 right triangle, so $DC = 4$. In addition, $\Delta DBC$ is similar to $\Delta ECF$ by AA Similarity. This tells us that $CE = 6$, $EF = 8$, and $CF = 10$. Using the Pythagorean Theorem on triangle $BCF$, we get $l = \boxed{5\sqrt{5}}$.

10. $\angle ACB$ is a right angle because it is inscribed in a semicircle.



Also, note that triangle $ABC$ is similar to triangle $ADE$. Using the Pythagorean Theorem, we find $AE = \sqrt{74}$. The ratio of $AB$ to $AE$ is $\frac{4}{\sqrt{74}}$, so the ratio of their areas will be $(\frac{4}{\sqrt{74}})^2 = \frac{16}{74}$. The area of triangle $ADE$ is $7 \cdot 5 \cdot \frac{1}{2} = \frac{35}{2}$, and let the area of triangle $ABC$ be $x$. We have the equation:

$$\frac{16}{74} = \frac{x}{\frac{35}{2}}$$

Solving this, we get $x = \boxed{\frac{140}{37}}$.

11. Let $BE = x$. Since the two smaller rectangles are congruent, $JK = BE = x$. Additionally, we have that $BC = EFF = KH = JG = 1$. Note that the triangles $FHK, DGH, AJG$, and $EKJ$ are similar by AA similarity. (To see this, first note that all are right triangles. Then, let angle $JKE = z$ and angle chase). Let $FH = y$. Then by Pythagorean theorem, we have $FK = \sqrt{1-y^2}$. Because triangle $EJK$ is similar to $FKH$, we have $\frac{KH}{HF} = \frac{JK}{KE}$, hence $EK = xy$. Since $EF = FK + KE = 1$, we get the equation $\sqrt{1-y^2} + xy = 1$. Again looking at triangle $EJK$ and its similarity to $FKH$, we see that $EJ = x\sqrt{1-y^2}$. Since triangle$GJA$ is similar to $KHF$, we see that $\frac{KH}{HF} = \frac{GJ}{JA}$, hence $JA = y$. So $BA = BE + EJ + JA = 1$, giving us $x + x\sqrt{1-y^2} + y = 1$.

    Now we must solve this system of equations: $\sqrt{1-y^2}+xy = 1$ and $x+x\sqrt{1-y^2}+y = 1$. We can rewrite these as $xy = 1 - \sqrt{1-y^2}$ and $x + y = 1 - x\sqrt{1-y^2}$. Squaring both equations, then subtracting the first from the second and rearranging, we get $4xy - 2x - 2y + 1 = 0$, or $(1-2x)(1-2y) = 0$. Hence, either $x = \frac{1}{2}$ or $y = \frac{1}{2}$. Trying both and plugging into the first equation, we see that when $x = \frac{1}{2}$, $y = 0$ and when $y = \frac{1}{2}$, $x = \boxed{2-\sqrt{3}}$, which is a valid solution.