Chapter 13

Coordinate Geometry

13.1 Warm-up Problems

1. Find the intersection of the lines $y = ax + b$ and $y = cx + d$ in terms of a, b, c, and d, given that they are not parallel.

2. Find the slope of the line $ax + by = c$ by rearranging the equation.

13.2 Coordinate Geometry Basics

To make solving a geometry problem more straightforward (and in many cases, merely a computational problem), we can plot the geometric figure on the coordinate axes. To illustrate this, let's start off with a simple example.

Example 13.1. Let $ABCD$ be a rectangle with $AB = 1$ and $BC = 2$. In addition, let E be a point on diagonal $\overline{BD}$ such that is trisects the diagonal and is closer to B than D. Find the length of segment $\overline{AE}$.

First, let's solve this problem with synthetic geometry (the "usual" kind of geometry without coordinates).

Checkpoint 13.1. Solve this problem. (Hint: Draw a parallel line to $\overline{AD}$ that passes through point E and use similarity).

Now we'll illustrate the technique of coordinate geometry.

Solution. We plot this rectangle on the coordinate plane with $A(0,0)$, $B(0,1)$, $C(2,1)$, and $D(2,0)$. Since E trisects $\overline{BD}$ and is closer to B than D, we have $E = \frac{2}{3}(0,1) + \frac{1}{3}(2,0) = \left(\frac{2}{3}, \frac{2}{3}\right)$. Hence, using the Distance Formula or using a 45-45-90 triangle, we see that $AD = \boxed{\dfrac{2\sqrt{2}}{3}}$. △

For simple problems, coordinate geometry (also known as analytical geometry) may not be the way to go since it can be computationally intensive. However, for harder problems with more complicated diagrams, you may not be sure what geometry technique to use, so coordinate geometry provides a very straightforward solution. Often, one of the most important steps in a coordinate geometry solution is choosing the location of the origin. Many times, it is a vertex or intersection of lines. It is helpful to choose the origin at a point that makes the expression for the final answer simple and easy to compute. In the previous example, we chose the origin to be at A (rather than B, C, or D) because calculating the distance from any point (in this case E) to the origin is straightforward.

To be able to use coordinate geometry effectively, it is necessary to be familiar with the following formulas (most of which you should have encountered before).

Definition. (Representations of a line) Recall that the following are representations of a line in the coordinate plane. Some forms lend themselves better in certain situations, so it is useful to know all of them.

- (Point-slope form) $y_2 - y_1 = m(x_2 - x_1)$
- (Standard form) $ax + by = c$
- (Slope-intercept form) $y = mx + b$

In addition, it is useful to know the following representation of a circle in the coordinate plane with center (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

Theorem 13.1. *Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 = -\frac{1}{m_2}$. In addition, these two lines are parallel if and only if $m_1 = m_2$.*

Checkpoint 13.2. Prove that two lines are parallel if and only if $m_1 = m_2$.

Theorem 13.2. *(Distance Formula) The distance between points (x_1, y_1) and (x_2, y_2) in the coordinate plane is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.*

Checkpoint 13.3. Prove the Distance Formula by drawing the auxiliary point (x_2, y_1) and forming the right triangle with vertices (x_1, y_1), (x_2, y_1), and (x_2, y_2).

Checkpoint 13.4. Prove that if two lines with slopes m_1 and m_2 are perpendicular, then $m_1 = -\frac{1}{m_2}$.

Theorem 13.3. *(Midpoint Theorem) The midpoint of the segment with endpoints (x_1, y_1) and (x_2, y_2) is the point with coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.*

Checkpoint 13.5. Prove the Midpoint Theorem.

To do this, verify that $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ lies on the line passing through (x_1, y_1) and (x_2, y_2) (using the point slope form) and that $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is equidistant from (x_1, y_1) and (x_2, y_2) (using the Distance Formula). Try it!

Theorem 13.4. *Let A and B be two points in the coordinate plane. Then the set of all points that are equidistant from A and B is the line perpendicular to $\overline{AB}$ that passes through the midpoint of $\overline{AB}$ - this line is called the perpendicular bisector of $\overline{AB}$.*

13.3 Fancy Coordinate Geometry

Theorem 13.5. *(Ratio Point Theorem) Let* $A = (x_1, y_1)$ *and* $B = (x_2, y_2)$. *The point*

$$P = (rx_1 + (1-r)x_2, ry_1 + (1-r)y_2)$$

lies on the line between A *and* B, *and splits the segment* AB *into a* $1-r : r$ *ratio, given* $0 < r < 1$. *If* $r < 1-r$, *the point* P *is closer to* B *than* A.

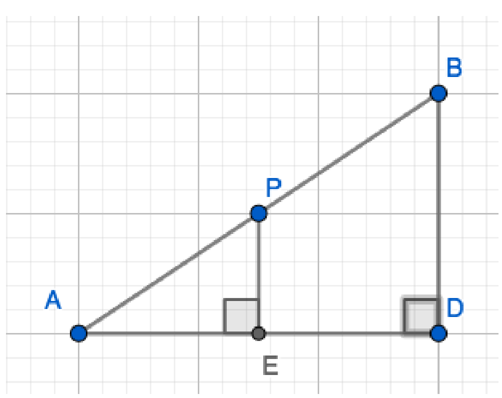
This is a more general version of the Midpoint Theorem (notice that if $r = \frac{1}{2}$, then it becomes the Midpoint Theorem). We'll use similarity to prove it:

Proof. First we'll prove that P lies on the line between A and B. Line AB in point slope form is $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$. Plugging in point P for x and y, we get

$$\begin{aligned}
(ry_1 + (1-r)y_2)) - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}((rx_1 + (1-r)x_2) - x_1) \\
\implies (r-1)y_1 + (1-r)y_2) &= \frac{y_2 - y_1}{x_2 - x_1}((r-1)x_1 + (1-r)x_2) \\
\implies (1-r)(y_2 - y_1) &= \frac{y_2 - y_1}{x_2 - x_1}(1-r)(x_2 - x_1) \\
\implies 1 &= \frac{y_2 - y_1}{x_2 - x_1} \cdot \frac{x_2 - x_1}{y_2 - y_1} \\
\implies 1 &= 1
\end{aligned}$$

Thus P is on line AB. To show that P is between A and B, note that $rx_1 + (1-r)x_2 = x_2 - rx_2 + rx_1$. Given that $0 < r < 1$, $x_2 - rx_2 + rx_1 < x_2 - (1)x_2 + (1)x_1 \implies x_2 - rx_2 + rx_1 < x_1$, and $x_2 - rx_2 + rx_1 > x_2 - (0)x_2 + (0)x_1 \implies x_2 - rx_2 + rx_1 > x_2$. This means the x-coordinate of P is between x_1 and x_2. We use a similar argument to show that the y-coordinate of P is between y_1 and y_2. As a result, P must be in between A and B.

WLOG, assume B is above and to the left of point A. Now we'll show that P splits AB into a $1-r : r$ ratio, using the diagram below:



Note that triangles AEP and ADB are similar by AA Similarity. Then the ratio of AP to AB will be equivalent to the ratio

$$\frac{AE}{AD} = \frac{rx_1 + (1-r)x_2 - x_1}{x_1 - x_2} = \frac{(r-1)(x_1 - x_2)}{(x_1 - x_2)} = r - 1$$

Now suppose $AP/PB = q \implies AP = qAB$. Then $PB = AB - qAB$, and $\frac{AP}{PB} = \frac{qAB}{AB - qAB} = \frac{q}{1-q}$. Substituting $r-1$ for q, we get $\frac{AP}{PB} = \frac{r-1}{1-(r-1)} = \frac{1-r}{r}$. Thus P splits AB into two segments with a ratio of $1 - r : r$. □

It can also be helpful sometimes to find the shortest distance from a point to a line.

Theorem 13.6. *(Distance from a Point to a Line) The distance between point $A(x_1, y_1)$ and the line $ax_2 + by_2 + c = 0$ is given by the expression*

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Moreover, the point on the line that is closest to A (called the projection of A onto the line) has coordinates

$$x = \frac{b(bx_1 - ay_1) - ac}{a^2 + b^2} \text{ and } y = \frac{a(-bx_1 + ay_1) - bc}{a^2 + b^2}.$$

This result can be proved many ways, such as by using similarity or the Distance Formula. However, all the various proofs require the construction of auxiliary lines, and these lines are either perpendicular to a coordinate axis or the line $ax_2 + by_2 + c = 0$ itself. We encourage you to try to work out the proof.

Theorem 13.7. *(Shoelace Theorem) The area of a n-sided polygon with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) listed in clockwise or counterclockwise order is given by*

$$\frac{1}{2}|(x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)|$$

This theorem is called the Shoelace Theorem because you list the vertices vertically:

$$\begin{array}{c}(x_1, y_1)\\(x_2, y_2)\\(x_3, y_3)\\\vdots\\(x_n, y_n)\\(x_1, y_1)\end{array}$$

By drawing "shoelaces" between the x and y coordinates, you can easily organize your computations.

There are many ways to prove this theorem. One way is to use induction by dividing the n-gon into a $(n-1)$-gon and a triangle, and use the Shoelace result for the area of a triangle (this can be proved by shifting the triangle to the origin and along the x-axis).

Checkpoint 13.6. Find the area of the triangle with vertices at $(2,2)$, $(5,4)$, and $(4,1)$.

Next, we will introduce the topic of lattice points and state some useful theorems regarding them.

Definition. A lattice point is a point in the coordinate plane with integer coordinates.

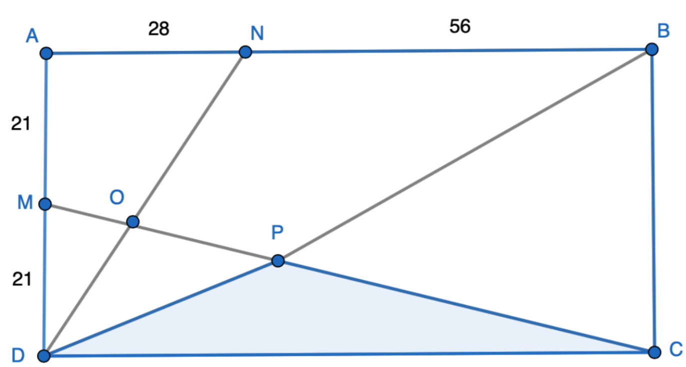
Theorem 13.8. *The number of boxes the segment from* $(0,0)$ *to* (m,n) *passes through is* $m+n-\gcd(m,n)$.

Theorem 13.9. *(Pick's Theorem) The area of a figure in the coordinate plane is given by* $\frac{b}{2}+i-1$*, where* b *is the number of lattice points on the border of the figure and* i *is the number of lattice points in the interior (excluding the border) of the figure.*

Now, to cap off our discussion of coordinate geometry, we'll solve the following example.

Example 13.2. Rectangle $ABCD$ has side lengths $AB=84$ and $AD=42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$. *Source: AIME*

Solution. Using the information given, we construct the following diagram.



Now, we impose a coordinate axes on this rectangle, with D being the origin and $\overline{CD}$ lying on the x-axis. Thus, we get can find the location of the following vertices easily: $A(0,42)$, $B(84,42)$, $C(84,0)$, $D(0,0)$, $M(0,21)$, and $N(28,42)$.

Next, we find the location of vertex O by using the fact that it lies on $\overline{MC}$ and $\overline{DN}$. Since we know the coordinates of M, C, D, and N, we see that $\overline{MC}$ is represented by the line $x+4y=84$ and $\overline{DN}$ is represented by the line $y=\frac{3}{2}x$. Plugging in the second equation into the first, we have $x+4\left(\frac{3}{2}x\right)=84$. Solving, we have $x=12$ and $y=18$. Hence, O is located at $(12,18)$.

Now, to find the coordinates of point P, we need to find the areas of $BNOC$ and ΔBPC. Using the Shoelace Theorem, we have $[BNOC]=2184$, so $[BPC]=1092$. Letting $P=(x,y)$, $b=BC$, and h be the height of ΔBPC from base $\overline{BC}$, we have

$[BPC] = 1092 = \frac{bh}{2} = \frac{42(84-x)}{2}$. Therefore, $x = 32$. Using the equation for the line containing $\overline{MC}$, we have that $y = 13$. Finally, we have that the area of ΔPCD (where h is the distance from P to $\overline{CD}$) is $\frac{CD \cdot h}{2} = \frac{84 \cdot 13}{2} = \boxed{546}$. △

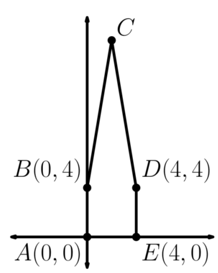
13.4 Exercises

1. Chris graphs the line $y = 3x + 7$ in the coordinate plane, while Sebastian graphs the line $y = ax + b$, for some numbers a and b. The x-intercept and y-intercept of Sebastian's line are double the x-intercept and y-intercept of Chris's line, respectively. What is the value of the sum $a + b$? *Source: MATHCOUNTS*

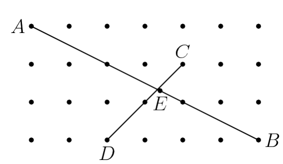
2. If x and y are integers such that $(x-3)^2 + (y+4)^2 = 25$, what is the greatest possible value of $x^2 + y^2$. *Source: AoPS*

3. When a triangle with vertices at $(2, 0)$, $(10, 2)$ and $(6, 6)$ is rotated 360 degrees about the point $(0, -3)$, the sides and vertices of the triangle sweep out a region in the shape of an annulus (ring). What is the area of the annulus? *Source: MATHCOUNTS*

4. Pentagon ABCDE has a vertical line of symmetry and has an area of 40 square units. How many lattice points are in the interior of the pentagon? *Source: AoPS*



5. The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE. *Source: Alcumus*



6. Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$? *Source: AMC 8*

7. In rectangle $ABCD$, we have $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ with $DF = 1$ and $GC = 2$, and lines AF and BG intersect at E. What is the area of ΔAEB? *Source: AMC 10*

8. Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on $\overline{AB}$ so that $EB = 1$, point G lies on $\overline{BC}$ so that $CG = 1$, and point F lies on $\overline{CD}$ so that $DF = 2$. Segments $\overline{AG}$ and $\overline{AC}$ intersect $\overline{EF}$ at Q and P, respectively. What is the value of $\frac{PQ}{EF}$? *Source: AMC 10*