Iowa City Math Circle Handouts

July 21, 2019

7 Probability and Expected Value

7.1 Warm-up Problems

1. When you roll a fair die, what is the probability that you will get an odd number?

2. When two fair dice are rolled, what is the probability that the two sides facing up sum to 10?

3. A fair coin is flipped four times. What is the probability it lands heads-up exactly two times?

4. A circle is inscribed in a square. If a random point inside the square is chosen, what is the probability that the point is also inside the circle?

5. Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, find the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left). *Source: AIME*

7.2 Probability

Probability allows us to assign a quantity to the likelihood that an event will occur. An event with probability 0 will not occur, and an event with probability 1 is certain to occur.

We can loosely define probability as the number of desired outcomes divided by the total number of outcomes. This means all of the concepts we have been talking about in counting also apply to probability, because all we need to do is to count the different numbers of outcomes. As a result, we can use constructive and complementary probability in the same way that we use constructive and complementary counting. Tools like combinations and permutations are also very helpful.

In probability, it's important to note that the probability of all outcomes must sum to 1. This means that if an event has a p probability of occurring, the probability of it **not** occurring is $1 - p$.

7.2.1 Exercises

1. You roll a die and then flip a coin. What is the probability that you get a number divisible by 3 on the die, and that you flip a heads?

2. Ten 6-sided dice are rolled. What's the probability that exactly one of the dice rolled is 1? *Source: Introduction to Counting and Probability*

3. I have a weighted coin that lands on heads with probability p. Suppose I flip my coin n times (n is a positive integer). Write a formula for the probability that exactly k of my flips come up heads. *Source: Introduction to Counting and Probability*

4. An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is \$20 or more? Assume that bills of the same value are indistinguishable. *Source: AMC*

5. Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same? *Source: AMC*

6. Four points, A, B, C, and D, are chosen randomly and independently on the circumference of a circle. What is the probability that segments AB and CD intersect? *Source: Alcumus*

7. If a, b and c are three (not necessarily different) numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, what is the probability that $ab + c$ is even? *Source: AHSME*

8. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N. *AIME*

9. A deck of forty cards consists of four 1's, four 2's,..., and four 10's. A matching pair (two cards with the same number) is removed from the deck. Given that these cards are not returned to the deck, find the probability that two randomly selected cards also form a pair. *Source: AIME*

10. Amelia has a coin that lands heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. Find the probability that Amelia wins. *AMC*

11. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where n is a positive integer. What is n? *Source: AMC*

12. For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6, on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6$. What is the probability of rolling a total of 7 on the two dice? *Source: AMC*

13. A box contains 3 shiny pennies and 4 dull pennies. One by one, pennies are drawn at random from the box and not replaced. What is the probability that it will take more than four draws until the third shiny penny appears? *Source: AHSME*

14. Tina randomly selects two **distinct** numbers from the set $\{1, 2, 3, 4, 5\}$, and Sergio randomly selects a number from the set $\{1, 2, ..., 10\}$. What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina? *Source: AMC*

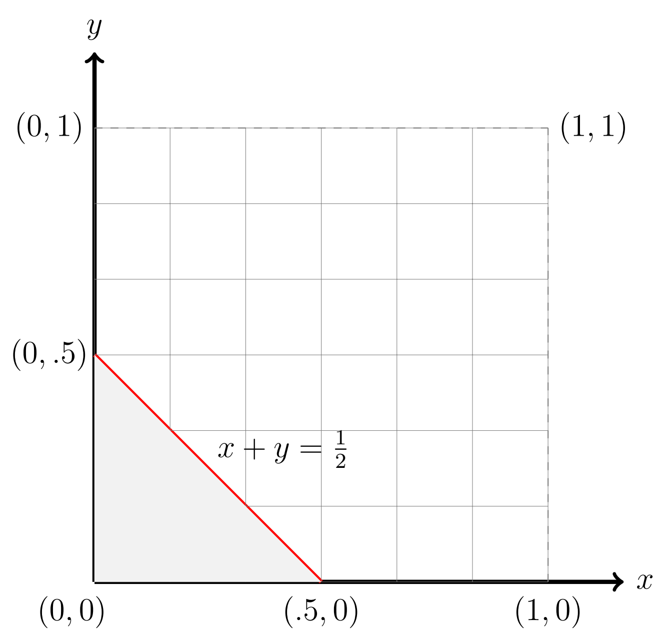
7.3 Geometric Probability

So far, we've been able to calculate the probability of an event occurring out of a finite number of outcomes. But what if there are an infinite number of outcomes?

Example 7.1. *Two real numbers are randomly chosen from between 0 and 1. What is the probability that their sum is less than* $\frac{1}{2}$*?*

In this problem, there are infinite possibilities for choosing two real numbers between 0 and 1, and there are an infinite number of ways for their sum to be less than $\frac{1}{2}$. So how can we solve this problem? We can use **geometric probability**. With geometric probability, we can measure the possible outcomes in terms of length, area, or volume.

Solution. The set of outcomes is the set of all (x, y) such that $x, y \in (0, 1)$. Note that this region is a 1×1 square, as shown in the figure below.

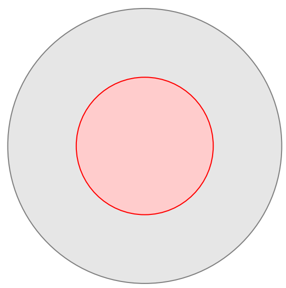


The set of all feasible outcomes (according to the problem statement) is the set of all (x, y) such that $x, y \in (0, 1)$ and $x + y < \frac{1}{2}$. The line $x + y = \frac{1}{2}$ is shown in red and the set of feasible outcomes is shaded in gray in the figure below. Now, to solve the problem, we must find the ratio of the area of the set of feasible outcomes (shown in gray) to the area of the total set of outcomes (the 1×1 square). Since the gray region is a triangle, its area is $\frac{1}{2} \cdot .5 \cdot .5 = \frac{1}{8}$. Since the total set of outcomes is a square with area 1, our answer is $\frac{\frac{1}{8}}{1} = \boxed{\frac{1}{8}}$. △

Now, let's look at another example of geometric probability. This time, our total set of outcomes is a circle.

Example 7.2. *A dart is thrown at a circular dartboard such that it will land randomly over the area of the dartboard. What is the probability that it lands closer to the center than to the edge? Source: Brilliant*

Solution. The total set of outcomes is the whole dart board (the gray circle in the figure below). The set of feasible outcomes is the red smaller circle inside the larger gray circle.

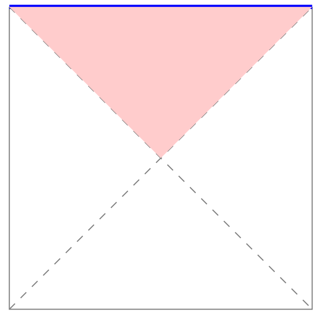


Notice that the circles have the same center and the radius of the smaller circle must be half the radius of the larger circle, so that all the points on the circumference of the smaller circle are equidistant from the center and the edge of the larger circle. Now, to solve the problem, we must find the ratio of the area of the set of feasible outcomes (the smaller, red circle) to the area of the total set of outcomes (the larger circle). Letting the radius of the smaller circle be r, the area of the smaller circle is πr^2 and the area of the larger circle is $\pi(2r)^2 = 4\pi r^2$. Hence, our desired probability is $\frac{\pi r^2}{4\pi r^2} = \boxed{\frac{1}{4}}$. △

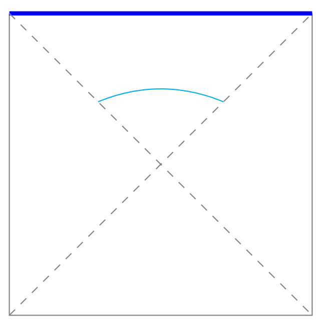
The next example is quite a bit more challenging, and the full solution of the problem requires calculus. However, understanding the general solution to the problem will be very useful, and only requires algebra.

Example 7.3. *A dart is thrown at a square dartboard such that it will land randomly over the area of the dartboard. What is the probability that it lands closer to the center than to the edge?*

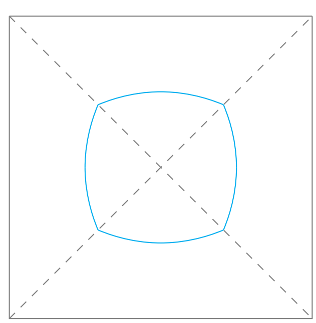
Solution. First, let's classify the points in the square closest to a specific edge. Since the points on a diagonal of the square are equidistant from the 2 sides the diagonal passes through, the set of points in the square equidistant from an edge form the right, isosceles triangle with right angle at the center of the square and the hypotenuse being the edge itself. This is depicted in the figure below, where the red triangle contains all the points in the square that are closer to the top edge (in blue) than any other edges.



Now, we aim to find the region of points in the square that are equidistant from the edge of the square and the center. To do this, we look at the points in the red triangle that are equidistant from the top side of the square (blue) and the center. Without loss of generality, let the square dartboard be 2×2, and let's fix the center of the square to be the origin in the Cartesian plane. We draw the coordinate axes parallel to the sides of the square. Let (x, y) be a point in the red triangle equidistant from the top side and the center. Then, we have that $\sqrt{x^2 + y^2} = 1 - y$, where $\sqrt{x^2 + y^2}$ is the point's distance to the center (which we designated as the origin) and $1 - y$ is the distance to the top side (which is part of the line $y = 1$). Squaring both sides and canceling the y^2 terms on both sides, we have that $y = -\frac{x^2}{2} + \frac{1}{2}$. Notice that this is just a parabola opening downward. Graphing this, we get the desired set of points closest to the top edge and equidistant from the top edge and the center.



We can similarly find this parabolic curve for each edge (we can redo the algebra or we can simply reflect the curve across the lines $y = x$, $y = -x$, $x = 0$, and $y = 0$), to get the set of all points in the square equidistant from the edge to the center (graphed below). Notice that the cyan parabolic curves don't form a circle, and instead form a *parabolic square*.



To complete the problem, we must find the area enclosed by the cyan curves using integration (a calculus technique). This is beyond the scope of this handout, but we encourage you to try it on your own if you know calculus. With integration, we get that the area enclosed is $\frac{4(4\sqrt{2}-5)}{3}$. Since the area of the square dartboard is $2^2 = 4$, the probability that the dart lands closer to the center than to the edge is $\frac{\frac{4(4\sqrt{2}-5)}{3}}{4} = \boxed{\frac{(4\sqrt{2}-5)}{3}}$. △

7.3.1 Review Exercises

1. Your bus is coming at a random time between 12 pm and 1 pm. If you show up at 12:45 pm, how likely are you to catch the bus? *Source: Brilliant*

2. Chloé chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloé's number? (Assume they cannot be equal) *Source: AMC*

3. Two real numbers are randomly chosen between 0 and 1. What is the probability that they are more than $\frac{1}{2}$ apart?

4. Allen and Bethany each arrive at a party at a random time between 1:00 and 2:00. Each stays for 15 minutes, then leaves. What is the probability that Allen and Bethany see each other at the party? *Source: Alcumus*

5. Two mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%,. Find m. *Source: AoPS*

6. A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Find the probability that the circle will not touch diagonal AC. *Source: AIME*

7.4 Conditional Probability

Let A and B be two events. Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where $P(A|B)$ denotes the probability that event A occurs given that event B occurs. Hence, we can read the above statement as "the probability that event A occurs given that event B occurs is equal to the probability that both A and B occur divided by the probability that B occurs". Thus, $P(A|B)$ is a *conditional probability.* This result may seem trivial, but it frequently comes up in competition math.

Example 7.4. *Given that you rolled an odd number when rolling a standard die, what is the probability that you rolled a multiple of* 3*?*

Solution. Let A be the event of rolling a multiple of 3 and let B be the event of rolling an odd number. Then we must find $P(A|B)$. Using our definitions, $A \cap B$ is the probability of rolling a 3 (which is $\frac{1}{6}$), and the probability of rolling an odd number is $\frac{1}{2}$. Hence, $P(A|B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \boxed{\frac{1}{3}}$. △

In essence, conditional probability problems can be solved by solving two probability problems. As a result, these problems are quite common but also can be harder and more tedious than other probability problems.

7.4.1 Review Exercises

1. Suppose you toss a coin 5 times. What is the probability that you toss at least three heads, given that you know at least one head has been tossed?

2. Suppose you toss a coin 5 times. What is the probability that you toss at least three heads, given that you know at least one *tail* has been tossed?

3. Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine? *Source: AMC 10*

4. Arnold is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of men. For each of the three factors, the probability that a randomly selected man in the population has only this risk factor (and none of the others) is 0.1. For any two of the three factors, the probability that a randomly selected man has exactly these two risk factors (but not the third) is 0.14. The probability that a randomly selected man has all three risk factors, given that he has A and B is $\frac{1}{3}$. Find the probability that a man has none of the three risk factors given that he does not have risk factor A. *Source: AIME*

7.5 Expected Value

Some events have a value associated with them. For example, rolling a die can produce the values of 1 through 6. So, given the probability of each event occurring, we can calculate the expected value of an event space. The expected value is another way of saying the "average" or "mean." In formal language, we have

$$E(X) = \sum_{x \in X} x \cdot P(x)$$

So to find the expected value, we must find the probability of each possible outcome.

Example 7.5. *What is the expected value of rolling a fair die?*

Solution. First, we must find the outcomes of rolling a die. They are obviously 1, $2, \ldots 6$, as these are the faces of a die. Now, we must find the probability of each outcome. Since we are rolling a fair die, each face of the die has a $\frac{1}{6}$ probability of landing face-up. So using the expression above, we have the expected value is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \boxed{\frac{7}{2}}$$

△

Expected value can often get very confusing. However, the answer is usually simpler than what you might expect (pun definitely intended).

7.5.1 Review Exercises

1. A carnival game involves flipping a fair coin. If it lands heads, you win \$50. If it lands tails, you lose \$30. You pay \$10 to play the game. What's the expected amount of money you gain or lose? Is this a fair game?

2. What is the expected value of the sum of the two faces when rolling a pair of die?

3. One million people buy a 1 dollar lottery ticket. The winner of the lottery wins 100 million dollars. Given that exactly one person wins, and that everyone has an equal chance of winning, what is the expected earning from entering the lottery?

4. Two standard dice are rolled. What is the expected number of 1's obtained? Express your answer as a common fraction. *Source: Alcumus*

5. Ben rolls two fair six-sided dice. What is the expected value of the larger of the two numbers rolled? Express your answer as a fraction. (If the two numbers are the same, we take that number to be the "larger" number.) *Source: Alcumus*

6. Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers? *Source: Alcumus*

7. * Jim flips a penny (50/50 chance of heads or tails) until he gets heads. What is the expected number of times he will flip the coin?