Iowa City Math Circle Handouts

July 14, 2019

6 Principle of Inclusion and Exclusion (PIE) and Venn Diagrams

6.1 Warm-up Problems

1. An 18-slice pizza was made with only pepperoni and mushroom toppings, and every slice has at least one topping. Only ten slices have pepperoni, and exactly ten slices have mushrooms. How many slices have both pepperoni and mushrooms? *Source: Alcumus*

2. Three-fourths of the students in Mr. Shearer's class have brown hair and six-sevenths of his students are right-handed. If Mr. Shearer's class has 28 students, what is the smallest possible number of students that could be both right-handed and have brown hair? *Source: Alcumus*

6.2 Basic Set Theory

A set is a collection of distinct objects. These objects can be anything; numbers, books, and people are all things that can be part of a set.

There are some notations we need know in order to understand PIE (let A be a set):

- We'll often enumerate the elements of sets by listing them between curly braces. For example, if A is the set of the first four positive integers, we might say that $A = \{1, 2, 3, 4\}$.

- If an object x is in set A, then we can say $x \in A$. Likewise, if x is not in A then $x \notin A$.

- $|A|$ is the number of objects in A. For example, if $A = \{1, 2, 3, 4\}$, then $|A| = 4$ because there are 4 objects in A.

- The *intersection* of sets A and B is the set of objects that are in **both** A and B. We represent the intersection of A and B as $A \cap B$. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \cap B = \{3, 4\}$ because 3 and 4 are in both A and B.

- The *union* of sets A and B is the set of all objects that are in A and B. We represent the intersection of A and B as $A \cup B$. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6\}$. Note that items that are in both A and B can only be included once.

6.2.1 Review Exercises

1. Let A and B be two sets. $|A| = a$, $|B| = b$, and $|A \cap B| = n$. Find $|A \cup B|$ in terms of a, b, and n.

2. Let A and B be two sets. $|A| = a$, $|B| = b$, and $|A \cup B| = u$. Find $|A \cap B|$ in terms of a, b, and u.

6.3 Venn Diagrams

Let's dive right in with an example:

Example 6.1. *How many positive integers less than* 100 *are multiples of* 3 *or* 5*?*

Solution. First, how many positive multiples of 3 are there below 100? The multiples of 3 below 100 are $3, 6, 9 \ldots 99$. There are 33 numbers in this sequence, since $\frac{99}{3} = 33$. Similarly, there are 19 multiples of 5 below 100 (Be careful not to include 100 as a multiple of 5, as the problem states "less than 100"). So is our answer $33 + 19 = 52$?

Let's take a closer look. 15 certainly fits the criteria in the problem, but it is counted in *both* sequences, as it is a multiple of both 3 and 5. This means that 15 was counted twice! Similarly, all other multiples of 3 and 5 have been counted twice. Notice that multiples of 3 and 5 are just multiples of $lcm(3, 5) = 15$. So we must subtract the number of multiples of 15 from our original sum of 52. There are 6 multiples of 15 below 100, so our final answer is $52 - 6 = \boxed{46}$. △

Checkpoint 6.1. In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle?

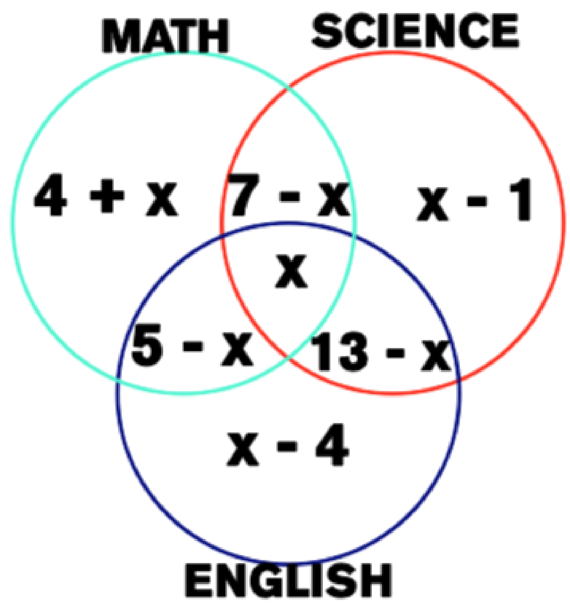
This reasoning might work fine for two sets. However, it's easier to draw a Venn diagram for more complex problems.

Example 6.2. *When Josh takes a survey of his 28 classmates, he gets the following results:*

- *14 like English class*
- *19 like Science class*
- *16 like Math class*
- *13 like English and Science class*
- *7 like Science and Math class*
- *5 like Math and English class*

If all classmates like at least one of English, Science, and Math class, how many students like English, Science, and Math?

Solution. Let x be the number of students who like English, Science, and Math. The goal is to find x. Using the information given in the problem, we can produce the following Venn Diagram.



We also know that the sum of all the entries in the above Venn diagram is equal to 28, since all 28 students liked at least one class. Therefore, we have $(4+x)+(7-x)+(5-x)+x+(x-1)+(13-x)+(x-4)=28$. Solving, we get that $x=\boxed{4}$. △

6.3.1 Review Exercises

1. How many positive integers less than or equal 500 are divisible by neither 2, 3, or 5? (Hint: try using complementary counting)

2. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren? *Source: AMC 8*

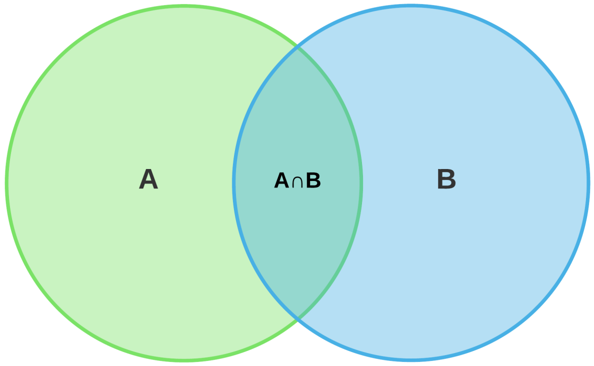
6.4 So what's PIE?

PIE is a technique used in counting and probability problems. In the previous example, you may have used PIE without even knowing it. Let's take a look at the statement of PIE itself.

For two sets A and B, we have

$$|A \cup B| = (|A| + |B|) - (|A \cap B|).$$

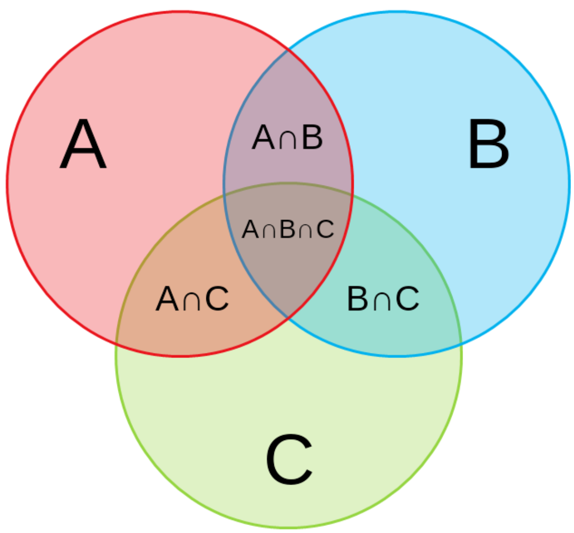
In the statement above, we are correcting for the double counting that occurs when we add the number of elements in A and B. This is shown intuitively in the Venn diagram below, where the intersection between the two circles represents $|A \cap B|$ and the union of the two circles (the entire colored region) represents $|A \cup B|$.



For three sets A, B, and C we have

$$|A \cup B \cup C| = (|A| + |B| + |C|) - (|A \cap B| + |B \cap C| + |C \cap A|) + (|A \cap B \cap C|).$$

For three sets, we need to correct for the double counting and triple counting that occur. As in the case for two sets, the statement for three sets can be easily derived from the Venn Diagram below.



Checkpoint 6.2. There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes? *Source: AMC 10*

For sets $A_1, A_2, \cdots, A_n$ in general, PIE says

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

Notice the placement of parentheses in the cases with 2 and 3 sets. This will help you generalize to cases with a higher number of sets. However, most competition problems involving PIE don't involve more than 4 sets.

Checkpoint 6.3. Try to draw a Venn diagram with 4 sets. Hint: get creative. You don't need to use circles to represent sets.

Checkpoint 6.4. Write the statement for PIE for 4 sets.

6.5 Problems

1. At Beaumont High School, there are 20 players on the basketball team. All 20 players are taking at least one of biology or chemistry. (Biology and chemistry are two different science courses at the school.) If there are 8 players taking biology and 4 players are taking both sciences, how many players are taking chemistry? *Source: Alcumus*

2. There are 30 students in Mrs. Taylor's kindergarten class. If there are twice as many students with blond hair as with blue eyes, 6 students with blond hair and blue eyes, and 3 students with neither blond hair nor blue eyes, how many students have blue eyes? *Source: Alcumus*

3. * How many 6 digit binary number have a string of 3 consecutive 1's in them? *Source: AoPS*

4. * Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, find the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left). *Source: AIME*

5. * The sequence 2, 3, 5, 6, 7, 10, 11, ... contains all the positive integers from least to greatest that are neither squares nor cubes nor perfect fifth powers (in the form of x^5, where x is an integer). What is the 1000^{th} term of the sequence?

6. * Six people of different heights are getting in line to buy donuts. Compute the number of ways they can arrange themselves in line such that no three consecutive people are in increasing order of height, from front to back. *Source: ARML*