Summer Math Circle Handouts

June 16, 2019

2 Solutions to Divisibility Problems

2.1 Warm-up Problems

1. $1001 = 7 \cdot 11 \cdot 13$

2. The least common multiple of 6 and 8 is 24, so the least number of packages of hot dogs Xanthia needs to buy is $24/6 = \boxed{4}$.

2.2 Prime Factorization Review Exercises

1. The prime factorization of 48 is $2^4 \cdot 3$. This means there are $5 \cdot 2 = 10$ factors of 48. To find the product of all of 48's factors, we can pair them up into 5 groups that multiply to 48: 1 and 48, 2 and 24, etc. Thus the product of the factors will be 48^5, so $n = 5$. The sum of the factors will be $(1+2+4+8+16)(1+3) = 124$. Thus $n + m = 5 + 124 = \boxed{129}$.

2.3 Modular Arithmetic Review Exercises

1. The LHS of $73 + 89 \equiv 3 + 9 \pmod{10}$ is equivalent to $3 + 9 \pmod{10}$ which is the same as the RHS, so this congruence is true.

2. We have the two congruences $a \equiv 7 \pmod{11}$ and $b \equiv 8 \pmod{11}$. Adding these two congruences together, we have $a + b \equiv 15 \equiv \boxed{4} \pmod{11}$.

3. $48^{48} \equiv (-1)^{48} \equiv \boxed{1} \pmod{7}$.

2.4 Sprint Exercises

1. To find the the least positive integer greater than 1 that leaves a remainder of 1 when divided by each of $2, 3, 4, 5, 6, 7, 8$ and 9, we can find the least common multiple of $2, 3, 4, 5, 6, 7, 8$ and 9 and add 1 to it. We find the prime factorizations of each of the numbers: $2, 3, 2^2, 5, 2 \cdot 3, 6, 2^3, 3^2$. The greatest power of 2 is 2^3, the greatest power of 3 is 3^2, and we also have a 5 and a 7. Thus the least common multiple will be $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. To get a remainder of 1, we add one to get $\boxed{2521}$.

2. $17 \equiv 3 \pmod{7}$ and $1234 \equiv 2 \pmod{7}$, so $17n \equiv 1234 \pmod{7}$ is equivalent to $3n \equiv 2 \pmod{7}$. $3 \cdot 3 \equiv 2 \pmod{7}$, so the smallest n will be $\boxed{3}$.

3. Because the numbers are easy to work with, you can just divide 123456 by 101 to get a remainder of $\boxed{34}$.

4. First we consider 3736. 3736 is divisible by 2, and $3736 \equiv 1 \pmod 3$. This means that $3736 \equiv 4 \pmod 6$ (Since it's even, we must have $1+3=4$). So $n \equiv -4 \pmod 6 \implies n = \boxed{2}$

5. For this problem it's easiest to make a table of all the values:

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	1	0	-1	-2	-1	0	1	2	1	0	-1
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
-2	-1	0	1	2	1	0	-1	-2	-1	0	1	2

Using the table, we get that "numeric" is equal to $-2+(-1)+(-1)+(-1)+2+1+1 = \boxed{-1}$.

6. We can split $1,234,567,890$ up into $12 \cdot 100^4 + 34 \cdot 100^3 + 56 \cdot 100^2 + 78 \cdot 100 + 90$. Since $100 \equiv 1 \pmod{99}$, this is equivalent to

$$12 \cdot 1^4 + 34 \cdot 1^3 + 56 \cdot 1^2 + 78 \cdot 1 + 90 \equiv 270 \equiv -27 \equiv \boxed{72} \pmod{99}$$

7. There are 21 terms in the sum (to verify this, add 4 to each term then divide by 5). Using the sum formula, the sum will be $(21)(1+101)/2 = 21 \cdot 51$. $21 \equiv 6 \pmod{15}$ and $51 \equiv 6 \pmod{15}$, so $21 \cdot 51 \equiv 6 \cdot 6 \equiv 6 \pmod{15} \implies n = \boxed{6}$.

8. From the warm up exercise, we know 1001 is a multiple of 7, so the smallest four-digit integer one less than a multiple of 7 will be $\boxed{1000}$.

9. $2004 = 2^2 \cdot 3 \cdot 167$. We want to make x, y and z as small as possible, so one of them has to be 167. For the remaining two variables, we can do 12 and 1, 2 and 6, or 3 and 4. 3 and 4 give the smallest sum, so the minimum possible value will be $3+4+167 = 174$.

10. There are 100 terms in the sum (add 1 to each term, then divide by 2). Using the sum formula, we have $100(1+199)/2 = 100 \cdot 100$. Since $100 \equiv 2 \pmod 7$, $100 \cdot 100 \equiv 2 \cdot 2 \equiv \boxed{4} \pmod 7$.

11. $10 \equiv 1 \pmod 9$, so taking the sum mod 9 is equivalent to adding up all the digits (why? We can write the numbers as $1 + (1 \cdot 10 + 2) + (1 \cdot 10^2 + 2 \cdot 10 + 3) + \cdots$). There are 8 1s, 7 2s, 6 3s, 5 4s, and so on. That means the sum is equivalent to $8 + 14 + 18 + 20 + 20 + 18 + 14 + 8 \equiv 2(-1+5+0+2) \equiv 12 \equiv \boxed{3} \pmod 9$.

12. We can divide both sides by $n!$ to get $(n+1) + (n+2)(n+1) = 440$. Factoring, we obtain $(n+1)(n+3) = 440$. Since n is an integer, $(n+1)$ and $(n+3)$ must be factors of 440. Note that 20 and 22 multiply to 440, and are two apart like $(n+1)$ and $(n+3)$. Thus $n+1 = 20 \implies n = \boxed{19}$.

13. If $x-3$ and $y+3$ are multiples of 7, then $x - 3 \equiv 0 \pmod 7 \implies x \equiv 3 \pmod 7$ and $y + 3 \equiv 0 \pmod 7 \implies y \equiv -3 \pmod 7$. That means $x^2 + xy + y^2 + n \equiv 3^2 + 3(-3) + (-3)^2 + n \equiv 9 + n \equiv 0 \pmod 7$. Thus the smallest n must be $\boxed{5}$.

14. For $\overline{24{,}z38}$ to be divisible by 6, it must be divisible by 2 and 3. We know it's divisible by 2 because it is even. To check divisibility by 3, we add up all the digits to get $2+4+z+3+8 = 17+z$. To make this sum a multiple of 3, z can be 1,4, or 7 for a sum of $1+4+7 = \boxed{12}$.

15. Since $5 \equiv 2 \pmod 3$, we can rewrite $5^n \equiv n^5 \pmod 3$ as $2^n \equiv n^5 \pmod 3$. Now it's a matter of guess and check. $2^1 \not\equiv 1^5 \pmod 3$, $2^2 \not\equiv 2^5 \pmod 3$, $2^3 \not\equiv 3^5 \pmod 3$, $2^4 \equiv 4^5 \pmod 3$, so the smallest n is $\boxed{4}$ (feel free to verify all of the (in)congruences on your own).

2.5 Problems

1. The greatest integer will be 4312, and the least will be 1324, for a total of $\boxed{5636}$.

2. We can use the formula from the handout to compute p quickly:

$$\sum_{i=1}^{\infty} \left\lfloor \frac{33}{3^i} \right\rfloor = \left\lfloor \frac{33}{3} \right\rfloor + \left\lfloor \frac{33}{9} \right\rfloor + \left\lfloor \frac{33}{27} \right\rfloor + \left\lfloor \frac{33}{81} \right\rfloor + \dots$$

This sums to $11+3+1 = \boxed{15}$.

3. For a number to be divisible by 44, it must be divisible by both 4 and 11. Using the divisibility rule for 11, we take the alternating sums of the digits of $\overline{5m5{,}62n}$ to get $5+5+2 = 12$ and $m+6+n$. For the number to be divisible by 11, $m+6+n$ can equal 12 or 23. So $m+n = 6$ or $m+n = 17$. Since we want the greatest number possible, we can let $m = 9$ and $n = 8$. 28 is divisible by 4, so this works. Thus the answer is $m+n = \boxed{17}$.

4. We can write out the terms of the Fibonacci sequence $\mod 4$, until the sequence starts to repeat. Let the terms in this new sequence be called $a_n = F_n \mod 4$. Calculating the first few terms of a_n, we get $1,1,2,3,1,0,1,1,2,3,1,0\dots$, which contains two full cycles of the sequence $\mod 4$ (we see that the sequence repeats after the first 6 terms). So our cycle is of length 6, which means the value of a_n is equivalent to $a_{n \mod 6}$. So a_{100} has the same value as a_4, because $100 \equiv 4 \mod 6$. Referring to the first 6 terms of a_n we calculated above, $a_4 = \boxed{3}$.

5. *Proof.* We can express every positive integer n with ones digit d_0, tens digit d_1, hundreds digit d_2, and so on as $d_0 + d_1 \cdot 10 + d_2 \cdot 10^2 + \cdots$. Since $10 \equiv 1 \pmod 3$ and mod 9, n is equivalent to $d_0 + d_1 + d_2 + \cdots$ in mod 3 and 9. In other words, if the sum of the digits of n are divisible by 3, then n is divisible by 3 (and the same for 9). □

6. The prime factorization of 4000 is $2^5 \cdot 5^3$. If n is positive, then the greatest possible power of 5 in the denominator of the fraction will be 3. If n is negative, 2 will be in the denominator of the fraction. The greatest possible power of 2 is 5. n can also be 0, so this gives us a total of $3+5+1 = \boxed{9}$ possible values of n.

7. We know that $n \equiv S(n) \mod 9$ from our divisibility rule for 9. So if $S(n) = 1274$, then $n \equiv S(n) \equiv 5 \mod 9$. So n has a remainder of 5 when divided by 9, which means that $n + 1 \equiv 6 \mod 9$. Using our divisibility rule from above, we have $n + 1 \equiv S(n + 1) \equiv 6 \mod 9$. So the answer must leave a remainder of 6 when divided by 9. The only answer choice that satisfies this is $\boxed{\textbf{(D)} 1239}$

8. It is easy to see that the smallest possible 4-digit integer with unique digits is 1234. Although 1234 and also 1235(the next smallest) is not divisible by its integers, $\boxed{1236}$ is because it is divisible by 2 and 3.

9. To find the remainder when divided by 45, we can find the remainders when divided by 5 and 9, which we know the divisibility rules for. $N \mod 5$ is equivalent to the unit digit of N, which is just 4. $N \mod 9$ is equivalent to the sum of the digits of N. However, instead of finding how many of each digit there is in N, we can simply calculate $1 + 2 + \ldots + 43 + 44$. This is because for each two-digit number, $xy \equiv x + y \mod 9$ (where xy represents the two-digit number $10x + y$, not $x \cdot y$). And for single-digit numbers, $x \equiv x \mod 9$ (where x represents a single digit). So $1 + 2 + \ldots + 43 + 44 = \frac{44 \cdot 45}{2} = 22 * 45 \equiv 0 \mod 9$, since 45 is divisible by 9. So we have that $N \equiv 4 \mod 5$ and $N \equiv 0 \mod 9$. So we must find a number between 0 and 44 that satisifes these properties. Testing multiples of 9, we see that 9 satisfies both conditions, so our answer is $\boxed{9}$.

10. We see that since QRS is divisible by 5, S must equal either 0 or 5, but it cannot equal 0, so $S = 5$. We notice that since PQR must be even, R must be either 2 or 4. However, when $R = 2$, we see that $T \equiv 2 \pmod 3$, which cannot happen because 2 and 5 are already used up; so $R = 4$. This gives $T \equiv 3 \pmod 4$, meaning $T = 3$. Now, we see that Q could be either 1 or 2, but 14 is not divisible by 4, but 24 is. This means that $R = 4$ and $P = \boxed{\textbf{(A)}\ 1}$. *Credit: AoPS*

11. We have $n = 100q + r$ by the definition of quotient and remainder. We must find values of n such that $q + r \equiv 0 \mod 11$. We can add $99q$ to both sides of the congruence, to get $100q + r \equiv 99q \mod 11$. However, since 99 is divisible by 11, $99q \equiv 0 \mod 11$. Also note that $n = 100q + r$. So substituting these values into the congruence $100q + r \equiv 99q \mod 11$, we get $n \cong 0 \mod 11$. So the problem is reduced to finding the number of 5-digit numbers that are divisible by 11 (it was given that n is a 5-digit number). The smallest 5-digit number divisible by 11 is $10010 = 910 \cdot 11$, and the largest is $99990 = 9090 * 11$ (this is easy to see from the fact that $9999 = 909 * 11$ is divisible by 11). Hence, the number of 5-digit numbers that are divisible by 11 is $9090 - 910 + 1 = \boxed{8181}$.

12. Note that $abc + ab + a = a(bc + b + 1)$. Also note that there is an equal number of numbers in the set that are 0, 1, and 2 $\mod 3$ (so there is a $\frac{1}{3}$ that a variable is a certain number $\mod 3$). This is because 2010 is divisible by 3. If $a \equiv 0 \mod 3$, then the expression will be divisible by 3 (since $3k$ is divisible by 3 for any integer k). The chance that a is $0 \mod 3$ is $\frac{1}{3}$.

We must now consider the case that a is not divisible by 3, or $a \equiv 1$ or $2 \mod 3$, which has a $\frac{2}{3}$ chance of happening. But now we must consider the value of $b \mod 3$. If $b \equiv 0 \mod 3$, then $0 \cdot c + 0 + 1 \equiv 1 \mod 3$, but this cannot happen as $bc + b + 1$ must be divisible by 3 (or $0 \mod 3$). Our next case is if $b \equiv 1 \mod 3$ (probability $\frac{1}{3}$), so $1 \cdot c + 1 + 1 \equiv c + 2 \equiv 0 \mod 3$, which means that $c \equiv 1 \mod 3$ (probability $\frac{1}{3}$). So the chance of this case happening is $\left(\frac{1}{3}\right)^2$.

Our next (and last) case is when $b \equiv 2 \mod 3$ (probability $\frac{1}{3}$). $2 \cdot c + 2 + 1 \equiv 2c \equiv 0 \mod 3$, so $c \equiv 0 \mod 3$ (probability $\frac{1}{3}$). The chance of this case happening is $\left(\frac{1}{3}\right)^2$. Combining all of our probabilities from all our cases, we get $\frac{1}{3} + \frac{2}{3}\left(\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \boxed{\frac{13}{27}}$.

13. Prime factorizing 323 gives you $17 \cdot 19$. The desired answer needs to be a multiple of 17 or 19, because if it is not a multiple of 17 or 19, the LCM, or the least possible value for n, will not be more than 4 digits. Looking at the answer choices, $\boxed{\text{(C) } 340}$ is the smallest number divisible by 17 or 19. Checking, we can see that n would be 6460.

14. As 71 is prime, c, d, and e must be 1, 1, and 71 (up to ordering). However, since c and e are divisors of 70 and 72 respectively, the only possibility is $(c, d, e) = (1, 71, 1)$. Now we are left with finding the number of solutions (a, b, f, g) satisfying $ab = 70$ and $fg = 72$, which separates easily into two subproblems. The number of positive integer solutions to $ab = 70$ simply equals the number of divisors of 70 (as we can choose a divisor for a, which uniquely determines b). As $70 = 2^1 \cdot 5^1 \cdot 7^1$, we have $d(70) = (1+1)(1+1)(1+1) = 8$ solutions. Similarly, $72 = 2^3 \cdot 3^2$, so $d(72) = 4 \times 3 = 12$.

    Then the answer is simply $8 \times 12 = \boxed{096}$. *Credit: scrabbler94 (AoPS)*

15. Let's assume that one of the children is 5 years old. Then 5 must divide the 4-digit license plate number, implying that the last digit of the 4-digit number must either be a 0 or a 5. Since the ages of the 8 children are 8 distinct integers from the set $\{1, 2, \ldots, 9\}$, we know that at least one child's age is even. Therefore, since the 4-digit number must be divisible by 2, its last digit cannot be 5 (the last digit must be even). Hence, the last digit must be 0.

    Since there are only 2 distinct letters in the 4 digit number and each digit is repeated twice, we know that the number must be of the form $dd00$ or $d0d0$, where d is an arbitrary digit. Additionally, we know that the last 2 digits of the number form Mr. Jones' age, so we can rule out the first case; thus, the number must be of the form $d0d0$.

    Now, we can write the numeral $d0d0$ as $d \cdot 1010 = d \cdot 101 \cdot 10$. We know that the number must be divisible by 3 since $3, 6$, and 9 are divisible by 3 and it also must be divisible by 4 since 4 and 8 are divisible by 4. Using the divisibility rule of 4, we must have the numeral $a0$ is divisible by 4 which means a is even. By the divisibility rule for 3, $a + a + 0 + 0$ should be divisible by 3 which means a is divisible by 3. Hence, we have that $a = 6$. However, using the divisibility rules for 7 and 9, 6060 is not divisible by 7 or 9. This is a contradiction, so $\boxed{5}$ cannot be the age of one of Mr. Jones' children.

16. So we must find the values of n such that $3^{n-1} + 5^{n-1} | 3^n + 5^n$. We know that

$$3^{n-1} + 5^{n-1} | (3^{n-1} + 5^{n-1}) \cdot (3 + 5) = 3^n + 5^n + 5 \cdot 3^{n-1} + 3 \cdot 5^{n-1}$$

Now we will use the property that if $k|a$ and $k|a+b$, then $k|b$. So we can subtract $3^n + 5^n$ from the right side of the above statement (using the very first statement, $3^{n-1} + 5^{n-1} | 3^n + 5^n$), to get

$$3^{n-1} + 5^{n-1} | 5 \cdot 3^{n-1} + 3 \cdot 5^{n-1} = 15(3^{n-2} + 5^{n-2})$$

so

$$3^{n-1} + 5^{n-1} | 15(3^{n-2} + 5^{n-2})$$

We will now take care of the case where $n = 1$. When $n = 1$, $3^{n-1} + 5^{n-1} = 2$, and $3^n + 5^n = 8$, and $2|8$. Hence, $n = 1$ satisfies the constraints. Now, for $n > 1$, we see that $\gcd(3^{n-1} + 5^{n-1}, 15) = 1$. This is because $3^{n-1} + 5^{n-1}$ is not divisible by 3 or 5, because $\gcd(3, 5) = 1$. Hence, for the above statement to be true, we must have

$$3^{n-1} + 5^{n-1} | 3^{n-2} + 5^{n-2}$$

However, if $a|b$ for positive a and b, then $a \leq b$. So we must have $3^{n-1} + 5^{n-1} < 3^{n-2} + 5^{n-2}$, which is not true for $n > 1$ because we have $3^{n-1} > 3^{n-2}$ and $5^{n-1} > 5^{n-2}$. Hence, there are not solutions for $n > 1$ and hence, the only solution is when $n = \boxed{1}$.