Summer Math Circle Handouts

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2 Divisibility

2.1 Warm-up problems

1. Factor 1001. (Remember this factorization, because it will be useful in many math competitions. And no, it is not prime.)

2. Xanthia buys hot dogs that come in packages of six, and she buys hot dog buns that come in packages of eight. What is the smallest number of hot dog packages she can buy in order to be able to buy an equal number of hot dogs and hot dog buns?

2.2 Divisibility Rules

Here is a quick refresher on several rules for deciding if a number is divisible by a certain number. It is worth learning these because they can often speed up computation by quite a bit.

- 3: If the sum of the digits of the number is divisible by 3, the number is divisible by 3.

- 5: If the last digit of the number is 0 or 5, the number is divisible by 5.

- 7: Subtract the two times the last digit from the number formed from the other digits. For example, for 238, $23 - 2 \cdot 8 = 7$. If the resulting difference is divisible by 7, the number is divisible by 7.

- 9: If the sum of the digits of the number is divisible by 9, the number is divisible by 9.

- 11: Add the alternating digits in the number to get two sums. For example, for 1232, the two sums are $1 + 3 = 4$ and $2 + 2 = 4$. If the difference of the two sums is divisible by 11, then the number is divisible by 11.

- 10^n: If the number ends in n zeros, the number is divisible by 10^n.

- 2^n: If the last n digits are divisible by 2^n, then the entire number is divisible by 2^n (Try to think about why this is). For example, a number is divisible by 8 if the last three digits are divisible by 8.

We encourage you to try to prove why all of these rules work on your own.

2.3 Prime Factorizations

Each number has a unique prime factorization. For example, 5's prime factorization is just 5 while the prime factorization of 60 is $2^2 * 3 * 5$. We can use prime factorizations to find the sum or product of all factors of a number, or the number of factors. If a number's prime factorization is written in the from

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_k^{e_k},$$

the total number of factors is

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

This means perfect squares have an odd number of factors. The sum of all factors for an integer is

$$\left(1 + p_1 + p_1^2 + \cdots + p_1^{e_1}\right)\left(1 + p_2 + p_2^2 + \cdots + p_2^{e_2}\right) \cdots \left(1 + p_k + p_k^2 + \cdots + p_k^{e_k}\right).$$

If the total number of factors of integer n is p, the product of all factors is $n^{p/2}$.

2.3.1 Review Exercises

1. The product of the factors of 48 can be written in the form 48^n and the sum of the factors of 48 is m. Compute $n + m$.

2.4 Factorials

An exclamation point indicates a factorial: $n!$ It means the product of all integers less than or equal to n. For example, 4! equals 24 or $4 \cdot 3 \cdot 2 \cdot 1$. Factorials often appear in many counting problems, but are also very helpful for number theory.

The number of zeroes at the end of a number (e.g. 1400 has 2 zeroes at the end of it) is equivalent to the maximum power of 10 that divides that number. For example, since 1400 has 2 zeroes at the end of it, 10^2 is the maximum power of 10 that divides 1400 (10^3 does not divide 1400). Below is a type of problem that commonly appears in math competitions.

Example 3. *Find the number of* 0*s at the end of* 7!

Solution. The prime factorization of 7! is $2^4 \cdot 3^2 \cdot 5 \cdot 7$. As mentioned above, the number of zeroes at the end of a number is equivalent to the maximum power of 10 that divides into that number. Let this number be n. This means that 10^n, or $2^n 5^n$ divides 7!. Hence, we can conclude that n is equal to the minimum of the exponents of 2 and 5 in the prime factorization. For the prime factorization of 7!, the exponent of 2 is 4 and the exponent of 5 is 1. So the minimum of 4 and 1, $\min(4, 1)$ is 1, hence there is 1 zero at the end of 7!. △

For factorials (greater than 1!), the exponent of 5 will always be less than the exponent of 2. Hence, we can simply calculate the exponent of 5 in the prime factorization of $n!$ to calculate the number of zeroes at the end of it.

To simplify this process even more, we don't have to find the prime factorization of the number! To find the number of 5's in the prime factorization of $n!$, we can use the following formula (Ask yourself why this works):

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{5^i} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \dots$$

We can now redo Example 3 much quicker using the formula above.
In general, to find the number of p's (or the exponent of p) in the prime factorization of $n!$, we can compute the following:

$$\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

2.5 Modular Arithmetic

We use the notation $a \equiv r \pmod{b}$ (which reads as "a is congruent to r mod b"), for integers a, b, and r, to mean that a leaves a remainder of r when divided by b. For example, $7 \equiv 3 \pmod{4}$ and $99 \equiv 1 \pmod{7}$. The following are fundamental rules of modular arithmetic that are derived from divisibility principles. In the following, let m be an integer such that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a, b, c, and d are integers.

1. Addition rule: $a + c \equiv b + d \pmod{m}$

2. Subtraction rule: $a - c \equiv b - d \pmod{m}$

3. Multiplication rule: $ac \equiv bd \pmod{m}$

4. Division Rule: $\frac{a}{e} \equiv \frac{b}{e} \pmod{\frac{m}{\gcd(m,e)}}$, where e is a positive integer that divides both a and b.

5. Exponentiation Rule: $a^e \equiv b^e \pmod{m}$ where e is a positive integer.

We will cover more applications of modular arithmetic, including linear congruences, the phi function, and Euler's theorem, in the next meeting.

2.5.1 Review Exercises

1. Is the following congruence true or false?

$$73 + 89 \equiv 3 + 9 \pmod{10}$$

   Source: AoPS Introduction to Number Theory

2. Let a, and b be positive integers who have remainders of 7 and 8 when divided by 11. What is the remainder of their sum when divided by 11?

3. Find the remainder when 48^{48} is divided by 7.

2.6 Sprint Exercises

The following exercises are not arranged in order of difficulty. It is worthwhile to do them consecutively in a short span of time to help improve your speed while solving competition problems. Many of the exercises below are from AoPS Online, Alcumus, and the AMC competitions.

1. What is the least positive integer greater than 1 that leaves a remainder of 1 when divided by each of $2, 3, 4, 5, 6, 7, 8$ and 9?

2. What is the smallest positive integer n such that $17n \equiv 1234 \pmod{7}$?

3. What is the remainder when 123456 is divided by 101.

4. Find the integer n, $0 \le n \le 5$, such that $n \equiv -3736 \pmod{6}$.

5. The letters of the alphabet are given numeric values based on the two conditions below.

   • Only the numeric values of -2, -1, 0, 1 and 2 are used.

   • Starting with A and going through Z, a numeric value is assigned to each letter according to the following pattern:

   $$1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 1, 0, -1, -2, -1, 0, \ldots$$

   Two complete cycles of the pattern are shown above. The letter A has a value of 1, B has a value of 2, F has a value of -2 and Z has a value of 2. What is the sum of the numeric values of the letters in the word "numeric"?

6. What is the remainder when $1, 234, 567, 890$ is divided by 99?

7. If
   $$1 + 6 + 11 + 16 + 21 + 26 + \cdots + 91 + 96 + 101 \equiv n \pmod{15},$$
   where $0 \le n < 15$, what is the value of n?

8. What is the smallest positive four-digit integer that is one less than a multiple of 7?

9. The product of positive integers x, y and z equals 2004. What is the minimum possible value of the sum $x + y + z$?

10. Find the modulo 7 remainder of the sum $1+3+5+7+9+\cdots+195+197+199$.

11. Find the remainder when
    $$1 + 12 + 123 + 1234 + 12345 + 123456 + 1234567 + 12345678$$
    is divided by 9.

12. A positive integer n satisfies the equation $(n+1)! + (n+2)! = 440 \cdot n!$. What is this integer?

13. Suppose $x-3$ and $y+3$ are multiples of 7. What is the smallest positive integer, n, for which $x^2 + xy + y^2 + n$ is a multiple of 7?

14. Determine the sum of all single-digit replacements for z such that the number 24,z38 is divisible by 6.

15. Determine the smallest positive integer n such that $5^n \equiv n^5 \pmod 3$.

2.7 Problems

Problems are roughly arranged in order of difficulty.

1. What is the sum of the least and the greatest positive four-digit multiples of 4 that can be written each using the digits 1, 2, 3 and 4 exactly once? *Source: AoPS*

2. What is the greatest integer p such that 33! has 3^p as a factor? *Source: MATHCOUNTS*

3. Given that m and n are digits, what is the sum of the values for m and n which yield the greatest six-digit number 5m5,62n that is divisible by 44?

4. The Fibonacci sequence is the sequence 1, 1, 2, 3, 5, ... where each term is the sum of the previous two terms. What is the remainder when the 100^{th} term of the sequence is divided by 4? *Source: AoPS*

5. Use modular arithmetic to prove the divisibility rules of 3 and 9.

6. For how many (not necessarily positive) integer values of n is the value of $4000 \cdot \left(\frac{2}{5}\right)^n$ an integer? *Source: AMC 10*

7. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$? *Source: AMC 10*

   (A) 1 **(B)** 3 **(C)** 12 **(D)** 1239 **(E)** 1265

8. What is the least four-digit positive integer, with all different digits, that is divisible by each of its digits?

9. Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45? *Source: AMC 10*

10. The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P? *Source: AMC 8*

11. Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11? *Source: AMC 10/12*

12. Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \dots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3? *Source: AMC 10*

    (A) $\frac{1}{3}$ **(B)** $\frac{29}{81}$ **(C)** $\frac{31}{81}$ **(D)** $\frac{11}{27}$ **(E)** $\frac{13}{27}$

13. Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323? *Source: AMC 10/12*

    (A) 324 **(B)** 330 **(C)** 340 **(D)** 361 **(E)** 646

14. Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following system of equations:

    $$abc = 70, cde = 71, efg = 72$$

    Source: AIME

15. Mr. Jones has eight children of different ages. On a family trip his oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is not the age of one of Mr. Jones's children? *Source: AMC 10/12*

    (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

16. Find all positive integers n such that $3^{n-1} + 5^{n-1}$ is a divisor of $3^n + 5^n$. *Source: St. Petersburg*

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