Summer Math Circle Handouts

June 23, 2019

3 Solutions to Modular Arithmetic Problems

3.1 Warm-up Problems

1. Note that all of the numbers in series are one greater than a power of 7. This means that each element of the series is congruent to $1 \pmod 7$. We can find the number of elements in the series by adding 6 to each element and dividing by 7. This gives us the sequence $1, 2, 3, \cdots, 101$, which means there are 101 terms in the sequence. Therefore, the sum is congruent to $101 \cdot 1 \equiv 101 \equiv \boxed{3} \pmod 7$.

2. $19^{19} \equiv (-1)^{19} \equiv -1 \equiv \boxed{4} \pmod 5$

3. We can multiply all three congruences together to get $xyz \equiv 6 \pmod 9$.

3.2 Bases: Review Exercises

1. Manually: 111010_2 is equivalent to $2^1 + 2^3 + 2^4 + 2^5 = 58_{10}$. We then convert 58_{10} to base 4. The greatest power of 4 less than 58 is 16, which divides 58 three times with a remainder of 10:

   $$3\_\_$$

   The next power of 4 is 4, which divides 10 twice with a remainder of 2, which gives us

   $$\boxed{322_4}$$

   Shortcut: We form the groups 11, 10, and 10. 11_2 in base 10 is 3 which is also 3 in base 4, and 10_2 in base 10 is 2 which is also 2 in base 4. This gives us $\boxed{322_4}$.

3.3 Modular Inverses: Review Exercises

1. Note that $3 \cdot 3 \equiv 1 \pmod 8$. So the modular inverse of 3 mod 8 is $\boxed{3}$.

2. $997 \equiv (-3) \pmod{1000}$. We'll first find the modular inverse of 3 mod 1000. This means that for some number n, $3n \equiv 1 \pmod{1000}$.

   We can list out the first few numbers equivalent to 1 mod 1000: 1001, 2001, 3001, and so on. Notice that $2001 = 3 \cdot 667$, so $3 \cdot 667 \equiv 1 \pmod{1000}$. Thus 667 is the modular inverse of 3.

   If 667 is the modular inverse of 3, then -667 is the modular inverse of -3 (why? $(-667) \cdot (-3) = 667 \cdot 3$). So $-667 \equiv \boxed{333} \pmod{1000}$ is the modular inverse of 997 mod 1000.

3.3.1 Euler's Totient Function: Review Exercises

1. The prime factorization of 15 is $3 \cdot 5$. Using the formula, $\phi(15) = (3^1 - 3^0)(5^1 - 0) = 2 \cdot 4 = \boxed{8}$.

2. The prime factorization of 10^n is $2^n \cdot 5^n$. Using the formula, $\phi(10^n) = (2^n - 2^{n-1})(5^n - 5^{n-1})$ or $4 \cdot 10^{n-1}$ after simplification.

3.4 Euler's Theorem: Review Exercises

1. *Proof.* We know that if p is a prime, $\phi(p) = p - 1$. Then Euler's Theorem gives us $a^{p-1} \equiv 1 \pmod{p}$. □

3.5 Problems

1. Assume Phil buys x packs of hot dogs. That means out of the $10x$ hot dogs he buys, 4 will be left over when matched with packs of 8 buns. Using modular arithmetic, this is equivalent to saying $10x \equiv 4 \pmod{8} \implies 2x \equiv 4 \pmod{8}$. x can be 2, 6, 10, and so on. Out of all the possibilities, $\boxed{6}$ is the second smallest.

2. Let the smallest multiple of 23 we are looking for be $23x$, where x is an integer. Then we have $23x \equiv 4 \pmod{89}$. Multiplying both sides by 4, we get $92x \equiv 16 \pmod{89}$. This is equivalent to $3x \equiv 16 \pmod{89}$. Now we multiply both sides by 30 to get $90x \equiv 480 \pmod{89}$, or $x \equiv 480 \equiv 35 \pmod{89}$. Thus the smallest multiple of 23 will be $23 \cdot 35 = \boxed{805}$.

3. Assume each bag initially contains x gold coins. After adding 53 coins, the total amount is divisible by 8, so $7x + 53 \equiv 0 \pmod{8}$. This is equivalent to $-x + 5 \equiv 0 \pmod{8}$, or $x - 5 \equiv 0 \pmod{8}$. Then $x \equiv 5 \pmod{8}$, so x could be 5, 13, 21, 29, and so on. $x = \boxed{29}$ is the smallest value that gives a total number of coins greater than 200.

4. $((((7)^7)^7)^{\cdots})^7$ is equivalent to $7^{7^{1000}}$. The unit digits of powers of 7 repeat in cycles of 4:

$$
\begin{aligned}
7^1 &\equiv 7 \pmod{10}\\
7^2 &\equiv 9 \pmod{10}\\
7^3 &\equiv 3 \pmod{10}\\
7^4 &\equiv 1 \pmod{10}\\
7^5 &\equiv 7 \pmod{10}\\
&\vdots
\end{aligned}
$$

   Thus it suffices to find the exponent, $7^{1000} \pmod{1000}$. We know $7^4 \equiv 1 \pmod{10}$, so $7^{1000} \equiv (7^4)^{250} \equiv 1^{250} \equiv 1 \pmod{1000}$, so the last digit of $7^{7^{1000}}$ will be $\boxed{7}$.

5. We consider all the elements of the set mod 7. This means we have 7 elements congruent to 0 mod 7, 8 elements congruent to 1 mod 7, and 7 elements congruent to 2 mod 7, 3 mod 7, 4 mod 7, 5 mod 7, and 6 mod 7. In order for any pair in S to not be divisible by 7, for any two elements a and b, $a+b \not\equiv 0 \pmod 7 \implies a \not\equiv -b \pmod 7$. This means that a number equivalent to $x \pmod 7$ and a number equivalent to $-x \pmod 7$ cannot both be in S. Thus, to maximize the number of elements in S, we can have 1 element congruent to 0 mod 7, 8 elements congruent to 1 mod 7, and 14 elements congruent to 2 and 3 mod 7 for a total of $1+8+14 = \boxed{23}$.

6. We wish to find the hundreds digit of 2011^{2011}. To do this, we take the expression mod 1000. $2011^{2011} \equiv 11^{2011} \pmod{1000}$.

   Now, note that Euler's Theorem gives us $11^{\phi(1000)} \equiv 1 \pmod{1000} \implies 11^{400} \equiv 1 \pmod{1000}$. Thus $11^{2011} \equiv 11^{5\cdot 400} \cdot 11^{11} \equiv 11^{11} \pmod{1000}$.

   This expression is now small enough to evaluate manually, but we can also use the Binomial Theorem to simplify the calculations.

   $$11^{11} \equiv (1+10)^{11} \quad \equiv \binom{11}{0} \cdot 1 + \binom{11}{1} \cdot 10 + \binom{11}{2} \cdot 100 \equiv 1 + 110 + 5500 \quad \equiv 661$$

   This is because any term in the expansion with a factor of 1000 can be canceled out. This gives us the hundreds digit $\boxed{6}$. Note that we could have used the Binomial Theorem from the beginning without using Euler's Theorem.

7. We have that $N^2 - N = N(N-1) \equiv 0 \mod 10000$

   Thus, $N(N-1)$ must be divisible by both 5^4 and 2^4. Note, however, that if either N or $N-1$ has both a 5 and a 2 in its factorization, the other must end in either 1 or 9, which is impossible for a number that is divisible by either 2 or 5. Thus, one of them is divisible by $2^4 = 16$, and the other is divisible by $5^4 = 625$. Noting that $625 \equiv 1 \mod 16$, we see that 625 would work for N, except the thousands digit is 0. The other possibility is that N is a multiple of 16 and $N-1$ is a multiple of 625. In order for this to happen, $N-1$ must be congruent to -1 (mod 16). Since $625 \equiv 1 \mod 16$, we know that $15 * 625 = 9375 \equiv 15 \equiv -1 \mod 16$. Thus, $N-1 = 9375$, so $N = 9376$, and our answer is $\boxed{937}$. *Credit: AoPS Wiki*

   For alternate solutions, including a brute-force method, see

   https://artofproblemsolving.com/wiki/index.php/2014_AIME_I_Problems/Problem_8.